Cauchy's Integral Formula & Fundamental Theorem

Sem - VI

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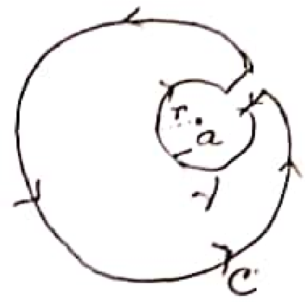
Cauchy's Integral Formula:

If $f(z)$ is analytic within and on a closed contour C and a is any point within C, then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z-a}$.

Proof: We describe a circle γ of radius r about a point $z=a$, lying entirely within C.

Consider the function $\frac{f(z)}{z-a}$. This function is analytic in the region between C and γ.

Hence by Cauchy's theorem for multi-connected region



We have
$$\int_C \frac{f(z)dz}{z-a} = \int_\gamma \frac{f(z)dz}{z-a}$$

$$\Rightarrow \int_C \frac{f(z)dz}{z-a} - \int_\gamma \frac{f(a)}{z-a}dz = \int_\gamma \frac{f(z)-f(a)}{z-a}dz$$

$$\Rightarrow \int_C \frac{f(z)dz}{z-a} - 2\pi i f(a) = \int_\gamma \frac{f(z)-f(a)}{z-a}dz \quad \left[\because \int_\gamma \frac{dz}{z-a} = 2\pi i\right]$$

$$\Rightarrow \left| \int_C \frac{f(z)dz}{z-a} - 2\pi i f(a) \right| = \left| \int_\gamma \frac{f(z)-f(a)}{z-a}dz \right|$$

$$\leq \int_\gamma \frac{|f(z)-f(a)|}{|z-a|}|dz|$$

$$< \epsilon \int_\gamma \left|\frac{dz}{z-a}\right| \quad \left[\because \text{Since } |f(z)-f(a)|<\epsilon \text{ Since } f(z) \text{ is continuous at } z=a\right]$$

$$= \frac{\epsilon}{r}\int_\gamma |dz| \quad \left[|z-a| = r \text{ for any } z \text{ on } \gamma\right]$$

$$= \frac{\epsilon}{r} \times 2\pi r$$

$$= 2\pi\epsilon \to 0 \text{ as } \epsilon \to 0.$$

Hence,
$$\int_C \frac{f(z)dz}{z-a} - 2\pi i f(a) = 0$$

$$\Rightarrow f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz.$$

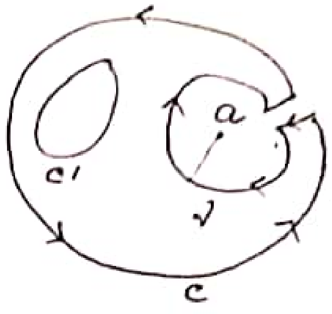
Cauchy's Integral formula for multiconnected regions:

If $f(z)$ is analytic in the region bounded by two closed curves c and c' and a is a point in the region, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z-a} - \frac{1}{2\pi i}\int_{c'} \frac{f(z)\,dz}{z-a}$$ where c is the outer contour.

Proof: We draw a small circle γ with its centre at the point a.

Now consider a function $\dfrac{f(z)}{z-a}$.



It is analytic in the region bounded by the three contours c, c' and γ.

Hence by Cauchy's theorem for multi-connected region,

$$\int_c \frac{f(z)\,dz}{z-a} = \int_{c'} \frac{f(z)\,dz}{z-a} + \int_{\gamma} \frac{f(z)\,dz}{z-a}$$

where integration round each curve is taken in such a way that the annular region always lies to the left.

Thus,
$$\int_c \frac{f(z)\,dz}{z-a} = \int_{c'} \frac{f(z)\,dz}{z-a} + 2\pi i f(a)$$ by Cauchy's formula.

$$\Rightarrow f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z-a} - \frac{1}{2\pi i}\int_{c'} \frac{f(z)\,dz}{z-a}$$

In general, if there be more curves c'', c''', ... then we have

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z-a} - \frac{1}{2\pi i}\int_{c'} \frac{f(z)\,dz}{z-a} - \frac{1}{2\pi i}\int_{c''} \frac{f(z)\,dz}{z-a} - \frac{1}{2\pi i}\int_{c'''} \frac{f(z)\,dz}{z-a} - \cdots$$

Cauchy's Integral formula for the derivative of an analytic function.

If a function $f(z)$ is analytic in a region D then its derivative at any point $z=a$ of D is also analytic in D, and is given by $f'(a) = \dfrac{1}{2\pi i}\displaystyle\int_c \frac{f(z)\,dz}{(z-a)^2}$ where c is any closed contour in D surrounding the point $z=a$.

Proof : Let $a+h$ be a point in the n.b.d. of the point a. Then by Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

$$f(a+h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a-h}\, dz$$

$$\therefore f(a+h) - f(a) = \frac{1}{2\pi i} \int_C \left[\frac{1}{z-a-h} - \frac{1}{z-a} \right] f(z)\, dz$$

$$= \frac{1}{2\pi i} \int_C \frac{h}{(z-a-h)(z-a)} f(z)\, dz$$

$$\Rightarrow \frac{f(a+h) - f(a)}{h} = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a-h)(z-a)}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a-h)(z-a)}$$

$$\Rightarrow f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^2}$$

As, a is any point of the region D, we conclude that $f'(a)$ is analytic in D. Hence, it is also established that the derivative of an analytic function is an analytic function of z.

Theorem: If a function $f(z)$ is analytic in a domain D, then $f(z)$ has at any point $z=a$ of D, derivatives of all orders, all of which are analytic function in D, their values are given by

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^{n+1}}$$

where C is any closed contour in D surrounding the point $z=a$.

Proof : This is proved by mathematical induction.

We have $f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^2}$

$\therefore$ So the theorem is proved for $n=1$.

Assume that, the theorem is true for $n=m$, m being positive integer.

$$\therefore \quad f^m(a) = \frac{m!}{2\pi i} \int_c \frac{f(z)\,dz}{(z-a)^{m+1}} .$$

Let $a+h$ be the point in the neighbourhood of the point $z=a$.

$$\therefore \quad f^m(a+h) = \frac{m!}{2\pi i} \int_c \frac{f(z)\,dz}{(z-a-h)^{m+1}}$$

Now,

$$\frac{f^m(a+h) - f^m(a)}{h} = \frac{m!}{2\pi i\, h}\left[\int_c \frac{f(z)\,dz}{(z-a-h)^{m+1}} - \int_c \frac{f(z)\,dz}{(z-a)^{m+1}}\right]$$

$$= \frac{m!}{2\pi i\, h} \int_c \left[\frac{1}{(z-a)^{m+1}} \left\{ \left(1 - \frac{h}{z-a}\right)^{-(m+1)} - 1 \right\}\right] f(z)\,dz$$

$$= \frac{m!}{h\, 2\pi i} \int_c \frac{1}{(z-a)^{m+1}} \left\{ (m+1)\frac{h}{z-a} + \frac{(m+1)(m+2)}{2!} \frac{h^2}{(z-a)^2} + \cdots \text{terms with higher powers of } h \right\} f(z)\,dz$$

Taking limit as $h \to 0$ we have

$$\lim_{h\to 0} \frac{f^m(a+h) - f^m(a)}{h} = \frac{m!\,(m+1)}{2\pi i} \int_c \frac{f(z)\,dz}{(z-a)^{m+2}}$$

$$\Rightarrow \quad f^{m+1}(a) = \frac{(m+1)!}{2\pi i} \int_c \frac{f(z)\,dz}{(z-a)^{m+2}}$$

This shows that the theorem is true for $n = m+1$ when it is true for $n = m$. By principle of mathematical induction, the theorem is true for any positive integer n.

Thus, $$f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)\,dz}{(z-a)^{n+1}}$$

Also we see from this result that $f^n(a)$ is an analytic function of z. This shows that derivatives of $f(z)$ of all orders are analytic if $f(z)$ is analytic.

<u>Converse of Cauchy's theorem</u> (~~or~~ Morera's theorem)

<u>Statement</u>: If $f(z)$ is a continuous function in a region D and if the integral $\int f(z)\,dz$ taken round any simple closed

contour in D, is zero then $f(z)$ is an analytic function inside D.

Proof: If z_0 be the fixed point and z be any variable point in the region D, Then the value of the integral $\int_{z_0}^{z} f(z)$ is independent of the curve joining z_0 and z.

So
$$F(z) = \int_{z_0}^{z} f(z)\,dz$$

$$\text{and } F(z+h) = \int_{z_0}^{z+h} f(z)\,dz.$$

$$\therefore F(z+h) - F(z) = \int_{z_0}^{z+h} f(z)\,dz - \int_{z_0}^{z} f(z)\,dz$$

$$= \int_{z}^{z+h} f(z)\,dz.$$

Let the points z and $z+h$ be on the closed curve C then

$$\lim_{h\to 0}\left[F(z+h) - F(z)\right] = \lim_{h\to 0} \int_{z}^{z+h} f(z)\,dz.$$

$$= \int_{C} f(z)\,dz \text{ where } C \text{ is the closed curve}$$

$= 0$ [As $h \to 0$, the two points coincide and the curve C becomes closed so that $\int_C f(z)\,dz = 0$]

$$\therefore \lim_{h\to 0}\left[F(z+h) - F(z)\right] = 0$$

$$\Rightarrow \lim_{h\to 0}\left[\frac{F(z+h) - F(z)}{h} - f(z)\right] = \lim_{h\to 0}\left(-\frac{f(z)}{h}\int_{z}^{z+h} dz\right).$$

$= \lim_{h\to 0} -\frac{f(z)}{h}\int_C dz$ [Since the curve C becomes closed as $h \to 0$]

$= 0$ [$\because \int_C dz = 0$]

$$\Rightarrow \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

$$\Rightarrow F'(z) = f(z)$$

We see that derivative of $F(z)$ exists for all values of z in D. Therefore, $F(z)$ is analytic in D. Consequently, $F'(z)$ i.e. $f(z)$ is also analytic in D. Because derivative of an analytic function is analytic.

Cauchy's Inequality:

If $f(z)$ is analytic within a circle C given by $|z-a| = R$ and if $|f(z)| \le M$ on C, then $|f^n(a)| \le \frac{M \lfloor n}{R^n}$

Ans:

We know that $f^n(a) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$

$$\therefore |f^n(a)| = \left| \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{n+1}} \right|$$

$$\le \frac{\lfloor n}{|2\pi i|} \int_C \frac{|f(z)||dz|}{|z-a|^{n+1}}$$

$$\le \frac{\lfloor n}{|2\pi i|} \frac{M}{R^{n+1}} \int_0^{2\pi} R\,d\theta$$

$$= \frac{\lfloor n\, M}{2\pi \cdot R^{n+1}} \times R\, 2\pi$$

$$= \frac{M \lfloor n}{R^n}$$

Since $z - a = Re^{i\theta}$
$dz = Re^{i\theta}.\, i d\theta$
$|dz| = R d\theta$.
& $|f(z)| \le M$

$\therefore |f^n(a)| \le \frac{M \lfloor n}{R^n}$. (proved).

Liouville's Theorem:

If a function $f(z)$ is analytic for all finite values of z and is bounded is a constant.

Proof: Since $f(z)$ is bounded so $|f(z)| \le M$, where M is a positive constant. Let us take two points z_1 and z_2 in z-plane.

Take a contour C to be a large circle, with its centre at origin and radius R enclosing the points z_1 and z_2.

So that $|z_1| < R$ and $|z_2| < R$.

We have by Cauchy's integral formula, we have

$$f(z_1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_1}\, dz$$

and $f(z_2) = \frac{1}{2\pi i} \int_c \frac{f(z)\,dz}{z - z_2}$

So that $f(z_1) - f(z_2) = \frac{1}{2\pi i} \int_c \frac{f(z)\,dz}{z - z_1} - \frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_2}\,dz$

$$= \frac{1}{2\pi i} \int_c \frac{z_1 - z_2}{(z - z_1)(z - z_2)} f(z)\,dz.$$

$\therefore \quad |f(z_1) - f(z_2)| = \left| \frac{1}{2\pi i} \int_c \frac{(z_1 - z_2) f(z)\,dz}{(z - z_1)(z - z_2)} \right|$

$$\leq \left| \frac{1}{2\pi i} \right| \int_c \frac{|(z_1 - z_2)|\,|f(z)|\,|dz|}{|z - z_1|\,|z - z_2|}$$

$$\leq \frac{1}{2\pi} |z_1 - z_2|\, M \int_c \frac{|dz|}{(|z| - |z_1|)(|z| - |z_2|)}$$

$$= \frac{1}{2\pi} \frac{|(z_1 - z_2)|\,M}{(R - |z_1|)(R - |z_2|)} \int_c |dz| \qquad [\because |f(z)| \leq M] \quad [\because |z| = R]$$

$$= \frac{|z_1 - z_2|\,M}{2\pi(R - |z_1|)(R - |z_2|)} \int_0^{2\pi} R\,d\theta \qquad z = Re^{i\theta} \Rightarrow |dz| = R\,d\theta$$

$$= \frac{|z_1 - z_2|\,M}{2\pi(R - |z_1|)(R - |z_2|)} \times 2\pi R \to 0 \text{ as } R \to \infty$$

Hence, $f(z_1) = f(z_2)$.

Since this holds for all values of z_1 and z_2. Therefore, $f(z)$ is constant.

<u>Note</u>: A function which is analytic in the whole of the z-plane is called an Integral function or <u>Entire function</u>.

$\boxed{\text{Ex}}$: Evaluate: (a) $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$

(b) $\oint_c \frac{e^{2z}}{(z+1)^4}\,dz$ where c is the circle $|z| = 3$.

<u>Ans</u>: (a) $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz -$

$$\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{z-1}\,dz \quad -(1)$$

By Cauchy's Integral formula, [Since $z=1$ and $z=2$ are inside c and $\sin\pi z^2+\cos\pi z^2$ is analytic inside c].

$$\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{z-2}\,dz = 2\pi i\left[\sin\pi 2^2+\cos\pi 2^2\right]$$
$$= 2\pi i$$

and
$$\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{z-1}\,dz = 2\pi i\left[\sin\pi 1^2+\cos\pi 1^2\right]$$
$$= -2\pi i$$

$\therefore$ From (1),
$$\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz = 2\pi i-(-2\pi i) = 4\pi i$$

(b) Here $f(z)=e^{2z}$.

By Cauchy's Integral formula, $f^{n}(a)=\dfrac{n!}{2\pi i}\oint_c \dfrac{f(z)}{(z-a)^{n+1}}\,dz$. $\quad -(1)$

Here $a=-1$, $n=3$, and $f'''(z)=8e^{2z}$.

Hence by (1), $f'''(-1)=\dfrac{3!}{2\pi i}\oint_c \dfrac{e^{2z}\,dz}{(z+1)^4}$.

$$\Rightarrow \oint_c \frac{e^{2z}}{(z+1)^4}\,dz = \frac{\cancel{2}\pi i\, f'''(-1)}{3\times\cancel{2}}$$
$$= \frac{\pi i}{3}\times 8e^{-2}$$
$$= \frac{8}{3}\pi i e^{-2}.$$

$\boxed{\text{Ex}}$: Evaluate $\dfrac{1}{2\pi i}\oint_c \dfrac{e^z}{z-2}\,dz$ if c is (a) the circle $|z|=3$
(b) the circle $|z|=1$.

Ans: (a) Here $f(z)=e^z$. $f(z)$ is analytic inside c and $z=2$ is inside c.

By Cauchy's integral formula,

$$\frac{1}{2\pi i}\oint_c \frac{e^z}{z-2}\,dz = f(2) = e^2$$
$$\Rightarrow \frac{1}{2\pi i}\oint_c \frac{e^z}{z-2}\,dz = e^2.$$

(b). $z=2$ is outside the circle $|z|=1$.

So $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = 0$ [By Cauchy's integral formula].

Fundamental Theorem of algebra:

Every non-constant polynomial with complex coefficients has at least one complex zero.

Proof: This theorem is proved by Liouville's theorem.

Let $p(z)$ be a polynomial of degree $n(\geq 1)$ where

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1}z^{n-1} + z^n \quad (a_0 \neq 0) \qquad (1)$$

To prove that $p(z)=0$ has a zero in $\mathbb{C}$.

This is proved by contradiction.

Suppose, $p(z)$ is not zero for any value of z. We have to prove that $p(z)$ is Bounded in $\mathbb{C}$.

For large z, we can expect that $p(z)$ should behave like z^n, since the largest power dominates the other ones.

Indeed, for $|z|\geq 1$ ie ($|z|^n \geq |z|^{n-1} \geq \cdots \geq |z|$), we have

$$|p(z)| = |a_0 + a_1 z + \cdots + a_n z^{n-1} + z^n|$$

$$= \left| z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots \frac{a_{n-1}}{z} + 1 \right) \right|$$

$$\geq |z|^n \left\{ 1 - \left| \frac{a_0}{z^n} + \cdots + \frac{a_{n-1}}{z} \right| \right\} \quad (\because \text{By triangle inequality})$$

$$\geq |z|^n \left\{ 1 - \left(\frac{|a_0|}{|z|^n} + \cdots + \frac{|a_{n-1}|}{|z|} \right) \right\}$$

$$\geq |z|^n \left[1 - \frac{1}{|z|} \left(|a_0| + \cdots + |a_{n-1}| \right) \right]$$

Hence, for sufficiently large $|z|$ ie. $|z| = R \geq R_0 = \max\{1, 2(|a_0| + \cdots |a_{n-1}|)\}$, we have

$$|p(z)| \geq \frac{|z|^n}{2}$$

Then for $|z| \geq R$, $\left| \frac{1}{p(z)} \right| \leq \frac{2}{|z|^n} \leq \frac{2}{R^n}$

On the set $\Delta_R = \{ z \in \mathbb{C} : |z| \leq R \}$, the function $\frac{1}{P(z)}$, being continuous on Δ_R, is bounded on the disk by some $M = \max_{|z|=R} \left| \frac{1}{P(z)} \right|$.

Therefore, $\left| \frac{1}{P(z)} \right|$ is bounded above for all $z \in \mathbb{C}$ by $\max\{M, \frac{2}{R^n}\}$.

Thus, $\frac{1}{P(z)}$ is bounded entire function and hence must be constant which is absurd (By Liouville's th) as $P(z)$ is not constant. Therefore, $P(z)=0$ has a zero.

Theorem: Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a non-constant polynomial of degree $n \geq 1$ with complex coefficients. Then p has n zeros in $\mathbb{C}$ i.e. there exists n complex numbers $z_1, z_2, \ldots, z_n$, not necessarily distinct such that $p(z) = a_n \prod_{k=1}^{n} (z - z_k)$.

Proof: If $a \in \mathbb{C}$, by division algorithm there is a polynomial q of degree $n-1$ such that $P(z) = (z-a)\, q(z) + R$ where R is constant. Clearly, $R = 0 \Leftrightarrow P(a) = 0 \Leftrightarrow (z-a)$ is a factor of $p(z)$.

Since there exists a z_1 such that $P(z_1) = 0$, $z - z_1$ is a factor of $P(z)$ with no remainder term. By division algorithm, $\exists$ a polynomial p_{n-1} of degree $n-1$ such that

$$p(z) = (z - z_1)\, p_{n-1}.$$

Because,

$$P(z) - P(z_1) = a_1(z - z_1) + \cdots + a_{n-1}(z^{n-1} - z_1^{n-1}) + (z^n - z_1^n)$$
$$= (z - z_1)\, p_{n-1}(z). \quad \left[\because P(z) = \sum_{k=0}^{n} a_k z^k \right]$$

This shows that p has a linear factor $(z - z_1)$. Thus, if $n > 1$, by applying the same principle, we conclude that there is another complex number say z_2 such that $p_{n-1}(z_2) = 0$ and so p_{n-1} has a linear factor $z - z_2$. Proceeding in this manner, we can express p uniquely as a product of linear factors $P(z) = a_n \prod_{k=1}^{n} (z - z_k)$, where $z_1, z_2, \ldots, z_n$ are (not necessarily distinct) the zeros of $P(z)$.

Rouche's Theorem :

If $f(z)$ and $g(z)$ are analytic with in and on a closed curve C and $|g(z)| < |f(z)|$ on C then $f(z)$ and $f(z)+g(z)$ have same number of zeros inside C.

Theorem. Every polynomial of degree n $(n \geq 1)$ has exactly n zeros (Using Rouche's theorem)

Proof : Suppose, the polynomial of degree n is

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0.$$

Choose, $f(z) = a_n z^n$ and $g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$.

If C is a circle having centre at the origin and radius $r > 1$.

Then on C,

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}|}{|a_n z^n|}$$

$$\leq \frac{|a_0| + |a_1| r + |a_2| r^2 + \cdots + |a_{n-1}| r^{n-1}}{|a_n| r^n}$$

$$\leq \frac{|a_0| r^{n-1} + |a_1| r^{n-1} + \cdots + |a_{n-1}| r^{n-1}}{|a_n| r^n}$$

$$= \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n| r}$$

Then by choosing r large enough we can make $\left|\frac{g(z)}{f(z)}\right| < 1$ on C

i.e. $|g(z)| < |f(z)|$, as $r > \dfrac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n|}$

Hence, by Rouche's Theorem, the given polynomial $f(z) + g(z)$ has the same number of zeros as $f(z) = a_n z^n$. But since, this last function has n zeros all located at $z=0$, $f(z) + g(z)$ also has n zeros.

Hence the result.

$\boxed{\text{Ex}}$ Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Ans : Consider the circle C_1: $|z| = 1$. Let $f(z) = 12$ and $g(z) = z^7 - 5z^3$. On C, we have $|g(z)| = |z^7 - 5z^3| \leq |z^7| + 5|z^3|$

$\leq 6 < 12 = |f(z)|$.

Hence by Rouché's theorem $f(z)+g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z|=1$ as $f(z)=12$, has no zeros inside C_1.

Consider circle $C_2: |z|=2$, let $f(z)=z^7$ and $g(z)=12-5z^3$.

On C_2, We have $|g(z)| = |12-5z^3| \leq 12 + |5z^3| \leq 60 < 2^7 = |f(z)|$

Hence by Rouché's theorem, $f(z)+g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z|=2$ as $f(z)=z^7$ i.e all the zeros are inside C_2.

Hence, all the roots lie inside $|z|=2$ but outside $|z|=1$.

Hence the problem.

$\boxed{\text{Ex}}$: Use Rouche's theorem, to show that $z^5+15z+1=0$ has one root in the disc $|z|<3/2$ and four roots in the annulus $\frac{3}{2}<|z|<2$.

Ans: Let $C_1: |z| = 2$ (circle)

Let $f(z)=z^5$ and $g(z)=15z+1$.

On C_1, $|g(z)| = |15z+1| \leq 15\times 2 + 1 = 31 < 2^5 = |f(z)|$ [$f(z)$ and $g(z)$ both are analytic within and on C_1].

Hence, by Rouche's Theorem, $f(z)+g(z) = z^5+15z+1$ has same number of zeros as $f(z)=z^5$ has.

But z^5 has five zeros all located inside $|z|=2$.

Hence, $z^5+15z+1$ also have all zeros inside $|z|=2$.

Consider circle $C_2: |z| = 3/2$.

Let $f(z)=15z$ and $g(z)=z^5+1$

Here $f(z)$ and $g(z)$ are both analytic within and on C_2.

and $|g(z)| = |z^5+1| \leq \left(\frac{3}{2}\right)^5+1 = \frac{275}{32} < 15\times\frac{3}{2} = \frac{45}{2} = |f(z)|$.

Hence by Rouché's theorem, $f(z)+g(z)$ has same number of zeros inside $|z|=3/2$ as $f(z)=15z$ has.

But $f(z)=15z$ has only one zero located inside $|z|=3/2$

Therefore, $f(z)+g(z) = z^5+15z+1$ has only one of its zeros inside $|z|=3/2$ and remaining four zeros must lie in the annulus $\frac{3}{2}<|z|<2$.