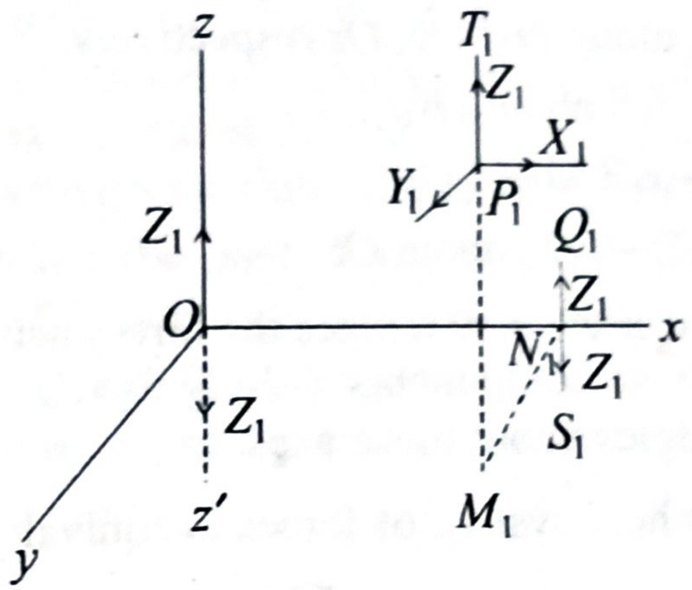
6.2. To find the resultant of any given system of forces acting at given points of a rigid body

Let any arbitrary chosen point O be taken as origin or base point and axes of co-ordinates Ox, Oy, Oz. Let $F_1, F_2, \ldots$ be a system of forces acting at different points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2) \ldots$ having the components (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), ... parallel to the axes.



At first consider the ferce F_1. Draw P_1M_1 perpendicular to the xy plane, M_1N_1 perpendicular to Ox and draw a line $Q_1N_1S_1$ parallel to Oz, so $ON_1 = x_1$, $N_1M_1 = y_1$ and $M_1P_1 = z_1$.

Introduce the forces Z_1 along N_1Q_1, N_1S_1, Oz and Oz'. These forces being two sets of equal and opposite forces do not alter the effect of given system of forces. Thus we have five equal forces acting parallel to axis of z.

Now the forces Z_1 along P_1T_1 and N_1S_1 form a couple whose moment is $Z_1 . M_1N_1 = y_1Z_1$, in a plane perpendicular to Ox and in the positive direction about Ox; so they are equivalent to a couple whose axis is along Ox and is positive.

The force Z_1 along N_1Q_1 and OZ' form another couple whose moment is $Z_1\, ON_1 = x_1Z_1$, in the plane perpendicular to Oy and in the negative direction about $O\,y$.

Hence the force Z_1 at P_1 is equivalent to: a force Z_1 at O along Oz, a couple of moment $(+y_1 Z_1)$ about Ox and a couple of moment $(-x_1 Z_1)$ about Oy.

Similarly the force X_1 at P_1 is equivalent to:

a force X_1 at O along Ox,

a couple of moment $+z_1 X_1$ about Oy

and a couple of moment $(-y_1 X_1)$ about Oz.

Again the force Y_1 at P_1 is equivalent to:

a force Y_1 at O along Oy,

a couple of moment $x_1 Y_1$ about OZ

and a couple of moment $(-z_1 Y_1)$ about Ox.

Hence finally the three component forces X_1, Y_1, Z_1 acting at P_1 (x_1, y_1, z_1) are equivalent to

forces X_1, Y_1, Z_1 along Ox, Oy, Oz respectively,

a couple $y_1 Z_1 - z_1 Y_1$ about Ox.

a couple $z_1 X_1 - x_1 Z_1$ about Oy

and a couple $x_1 Y_1 - y_1 X_1$ about Oz.

In a similar manner we may replace the forces acting at other points (x_2, y_2, z_2), . . .whose components (X_2, Y_2, Z_2), . . .by forces along Ox, Oy, Oz and couples about those axes.

Therefore, the whole system of forces is equivalent to

a force along $Ox = X_1 + X_2 + \cdots = \sum X_1 = X$,

a force along $Oy = Y_1 + Y_2 + \cdots = \sum Y_1 = Y$,

a force along $Oz = Z_1 + Z_2 + \cdots = \sum Z_1 = Z$,

a couple of moment $= \sum (y_1 Z_1 - z_1 Y_1)$ about $Ox = L$

a couple of moment $= \sum (z_1 X_1 - x_1 Z_1)$ about $Oy = M$,

and a couple of moment $= \sum (x_1 Y_1 - y_1 X_1)$ about $Oz = N$.

Again these three forces are equivalent to a single force R acting through O, such that $R^2 = X^2 + Y^2 + Z^2$ and the direction cosines of its line of action are $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$. Similarly the three couples of moments L, M, N are together equivalent to a single couple of moment G, such that

$G^2 = L^2 + M^2 + N^2$ and the direction cosines of its axis are $\frac{L}{G}, \frac{M}{G}, \frac{N}{G}$.

Hence the entire system of forces acting on a rigid body has been reduced to a single force R acting at arbitrary chosen point O together with a couple of moment G whose axis passes through O.

6.3. General conditions of equilibrium

The forces acting on body keep it in equilibrium if and only if

(i) the force R has no tendency to give translational displacement to the body so that

$$R = 0 \Rightarrow X = 0, Y = 0, Z = 0$$

(ii) Couple G has no tendency to give rotational displacement to the body so that

$$G = 0 \Rightarrow L = 0, M = 0, N = 0 .$$

Thus the necessary and sufficient conditions for the equilibrium are that $X = 0$, $Y = 0$, $Z = 0$, $L = 0$, $M = 0$, $N = 0$.

Note: The combination of a force and a couple is often called a *dyname* and the six quantities X, Y, Z, L, M, N are its components or elements of the system sometimes its denoted by $(X, Y, Z; L, M, N)$.

6.4. G changes when the base changes but R does not change

Let a system of forces acting on a rigid body can be reduced to a single force R acting through an arbitrary chosen point O and a couple of moment G whose axis passes through that point O where $R^2 = X^2 + Y^2 + Z^2, G^2 = L^2 + M^2 + N^2$.

Let $O'(\xi, \eta, \zeta)$ be any other base. To find the values of six components w.r.t. base O' we transfer the origin O to O'. Due to this changes co-ordinates of the point of application of the forces remains change, *viz.*, $(x_1 - \xi, y_1 - \eta, z_1 - \zeta)$ ... etc, but components (X_1, Y_1, Z_1) ... etc. remains unchanged. Hence X, Y, Z remains unchanged but L, M, N remains changes to L', M', N'

where $L' = \sum\{(y_1 - \eta)Z_1 - (z_1 - \zeta)Y_1\}$

$$= \sum(y_1 Z_1 - z_1 Y_1) - \eta\sum Z_1 + \zeta\sum Y_1$$

$$= L - \eta Z + \zeta Y,$$

$$M' = \sum\{(z_1 - \zeta)X_1 - (x_1 - \xi)Z_1\}$$

$$= \sum(z_1 X_1 - x_1 Z_1) - \zeta\sum X_1 + \xi\sum Z_1$$

$$= M - \zeta X + \xi Z .$$

$$N' = \sum\{(x_1 - \xi)Y_1 - (y_1 - \eta)X_1\}$$
$$= \sum(x_1 Y_1 - y_1 X_1) - \xi\sum Y_1 + \eta\sum X_1$$
$$= N - \xi Y + \eta X\ .$$

Thus L, M, N are changed and therefore G changes when the base changes but R does not change.

6.5. Definitions

Central axis: *If a system of forces acting on a rigid body be reduced to a single force together with a couple whose axis is along the direction of the force, then that line is called central axis.*

Wrench : *Suppose a system of forces is reduced to a single force R and a couple of moment* $\Gamma = G\cos\theta$ *whose axis coincides with the direction of acting force. Then* R *and* Γ *taken together are called wrench of the system and are written as* (R, Γ).

Pitch : *The ratio* $\frac{\Gamma}{R} = p$ *is called the pitch of the system and is of a linear magnitude.*

Intensity : *The single force R is called the intensity of the wrench.*

Screw : *The straight line along which the single force acts when considered together with the pitch, is called screw.* So, screw is a definite straight line associated with a definite pitch.

Note : When pitch, is zero then wrench reduces to a single force $(\Gamma = 0)$. When pitch is infinite then $R = 0$ hence wrench reduces to a cuple Γ only.

6.6. A given system of forces are acting on a rigid body can be reduced to a single force together with a couple whose axis coincide with the direction of the force.

Or, Every given system of forces acting on a rigid body can be reduced to a wrench.

We know that every system of forces acting on a rigid body can be reduced to a single force, say, R, acting at any point O along OA (say) together with a couple of moment G about a line OB (axis) through O. Let $\angle AOB = \theta$. Draw OC perpendicular to OA, in the plane of AOB, so that OA, OB and OC all in one plane AOB. Again draw OD perpendicular to the plane AOC (Fig.1).

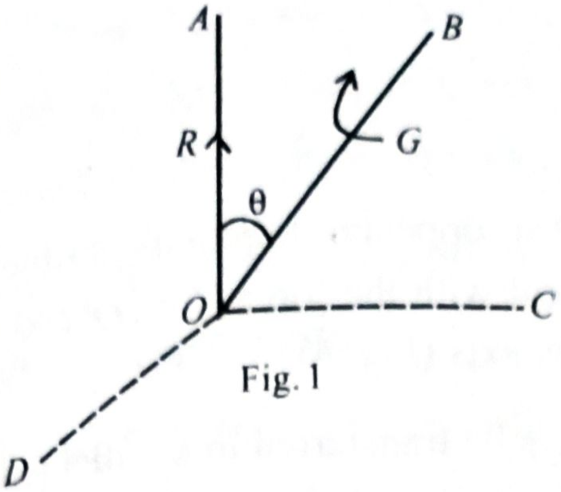


Fig. 1

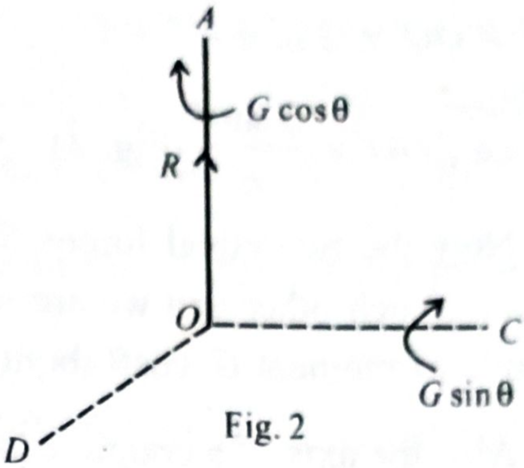


Fig. 2

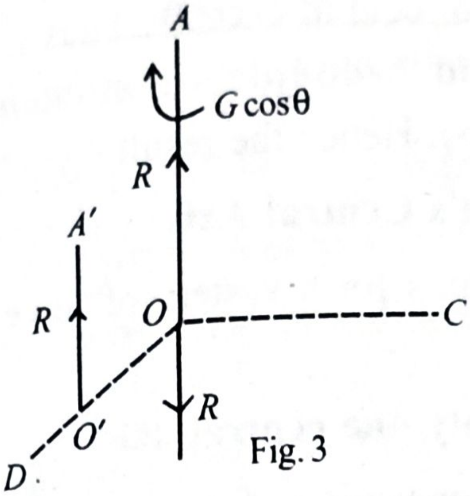


Fig. 3

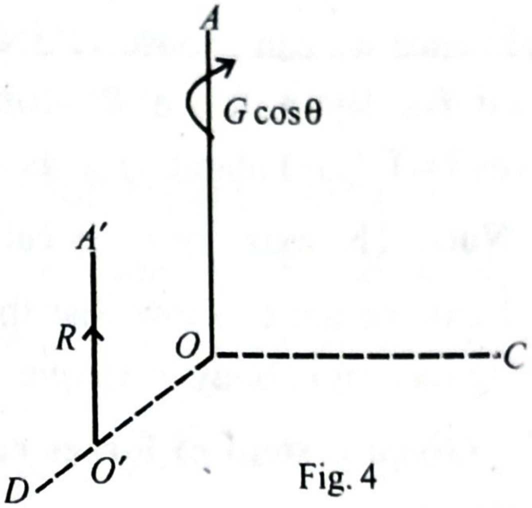


Fig. 4

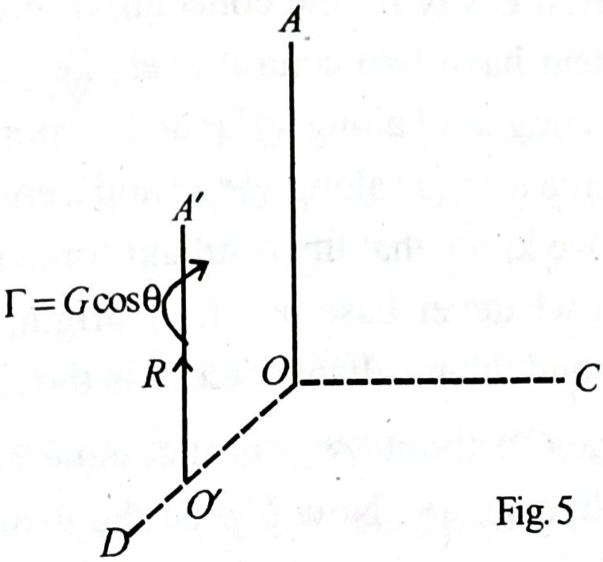


Fig. 5

Now the couple G about OB as axis is equivalent to a couple $G \cos\theta$ about OA as axis and a couple $G \sin\theta$ about OC as axis (Fig.2).

Again the couple $G \sin\theta$ acts in the plane AOD because its axis OC is perpendicular to the plane AOD. Hence the couple $G \sin\theta$ can be replaced by any two equal unlike parallel forces of moment $G \sin\theta$.

Let one of the forces be R acting at O in the direction opposite to OA. Then the other force must be equal to R acting parallel to OA at some point O', say, in OD, such that

$R.OO' = G \sin\theta$

i.e., $OO' = \dfrac{G \sin\theta}{R}$ (Fig. 3)

Now the two equal forces R at O in opposite directions, so they balance each other and we are now deal with the force R at O' and a couple of moment $G\cos\theta$ about OA as axis (Fig. 4).

Also the axis of a couple can always be transferred to parallel axis and hence we can choose $O'A'$ as the axis of couple $G\cos\theta$. Thus we have finally a force R along $O'A'$ and a couple of moment $G\cos\theta = \Gamma$ (say) about $O'A'$ as axis (Fig. 5). Hence the result.

Note: The axis, $O'A'$, is called **Poinsot's Central Axis.**

Next we are to show that this central axis for a system of forces acting on a rigid body is unique

6.7. Given system of forces can have only one central axis

We can prove this result easily by the conception of contradiction. Let, if possible, the system have two central axes, i.e., the system be equivalent to a force R acting at O along $O'A'$ and a couple Γ' about a line $O'A'$ and also to a force R at O'' along $O''A''$ and a couple Γ'' about another line $O''A''$. But we know that the resultant force is the same in magnitude and direction whatever base point, or origin, is taken. So, $O''A''$ is parallel to $O'A'$ and the resultant force R is the same for each.

Hence the system (R, G') about $O'A'$ is the same as the system (R, G'') about a parallel line $O''A''$. Now if p be the distance between $O'A'$ and $O''A''$ then R along $O''A''$ is equivalent to R along $O'A'$ with a couple of moment $R.p$ about an axis perpendicular to $O'A'$. Thus the second system is equivalent to a force R along $O'A'$, a couple G'' about $O'A'$ and a couple of moment $R.p$ about an axis perpendicular to $O'A'$, i.e., it is equivalent to a force R along $O'A'$ and a couple about on axis which is not $O'A'$, i.e., it is not equal to the system (R, G') with $O'A'$ as axis. Hence our original assumption is wrong, i.e., we can not find out two central axes $O'A'$ and $O''A''$, i.e., the central axis is unique.

Note: The moment of the resultant couple about the central axis is less than the moment of the resultant couple corresponding to any point which is not on the central axis. Because $G\cos\theta < G$ for $\cos\theta < 1$.

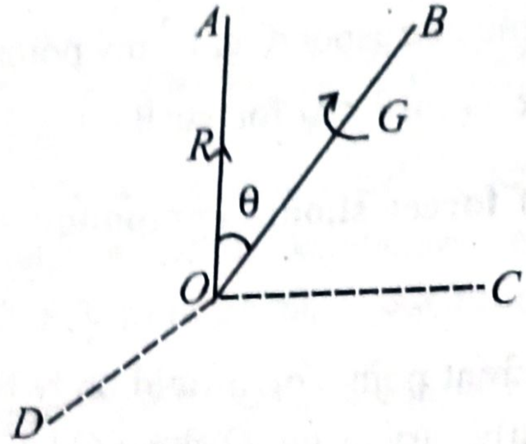
6.8. Conditions that a given system of forces should compounded into a single force.

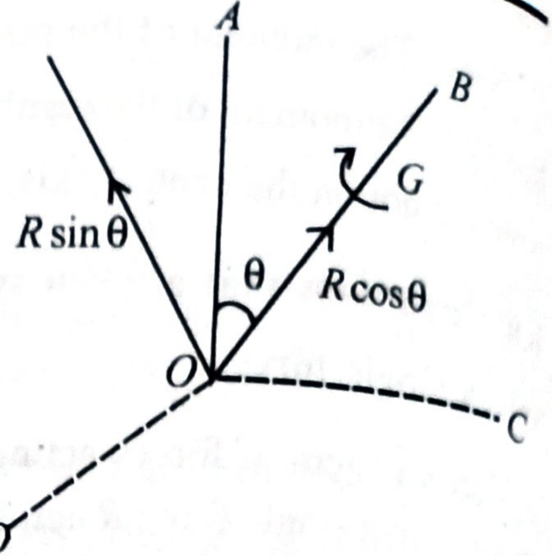
Let a system of forces acting at different points on a rigid body be reduced to a single force R acting at an arbitrary origin O along OA and a single couple of moment G about OB as an axis such that $\angle AOB = \theta$.

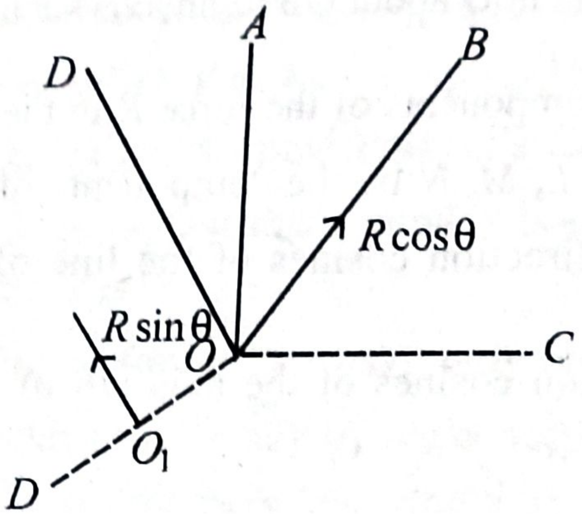
If X, Y, Z be the components of the force R in the directions of th co-ordinate axes and L, M, N be the components of G about the co ordinate axes then direction cosines of the line of action of R are $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ and direction cosines of the axis OB of the couple G are $\frac{L}{G}, \frac{M}{G}, \frac{N}{G}$. Therefore,

$$\cos\theta = \frac{X}{R}\cdot\frac{L}{G} + \frac{Y}{R}\cdot\frac{M}{G} + \frac{Z}{R}\cdot\frac{N}{G} = \frac{LX + MY + NZ}{RG} \quad \ldots (1)$$

Now the force R is equivalent to a force $R\cos\theta$ along the axis OB of the couple and a force $R\sin\theta$ along perpendicular to OB, i.e., in the plane of the couple G. Thus in the plane of the couple G there are three parallel forces: one $R\sin\theta$ and other two equal and unlike parallel forces which form the couple G. Again, since a single force and a couple are together equivalent to a single force which is parallel to the given force, so, force $R\sin\theta$, together with the parallel forces of the couple, are, equivalent to a parallel force $R\sin\theta$ which does not pass through O and therefore cannot, ingeneral, compound with $R\cos\theta$ into a single force.







Hence, in order that the system should reduce to a single force, we should have $\theta = \pi/2$ because that case $R\cos\theta = 0$ and then we are left with a single force $R \sin\theta$ not acting through O.

Hence, from (1), we get

$LX + MY + NZ = 0$, provided X, Y, Z do not all vanishes. Because, if $R = 0$ then the system reduces to a couple of moment G. But if $G = 0$, then the system reduces to a single force R and $LX + MY + NZ = 0$. Thus the conditions that a given system of forces should compounded into a single force are $LX + MY + NZ = 0$ and $X^2 + Y^2 + Z^2 \neq 0$ i.e., $R \neq 0$.

These are required conditions.

6.9. Invariants of the system

Whatever origin, or base point and axes are chosen, for any given system of forces the quantities $X^2 + Y^2 + Z^2$ and $LX + MY + NZ$ are invariable.

Let $F_1, F_2, \ldots$ be a system of forces acting at the different points $A_1, A_2 \ldots$ on a rigid body. Let O be any arbitrary chosen point of the body. Let us consider O as the origin and any three mutually perpendicular lines Ox, Oy, Oz as the rectangular axes. Let (x_1, y_1, z_1) (x_2, y_2, z_2). . .be the co-ordinates of A_1, A_2. . .and let $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2)$, . . .be the components of the forces F_1, F_2. . .and (l_1, m_1, n_1), (l_2, m_2, n_2), . . .be the direction cosines of line of action of F_1, F_2,. . .

Now, we know that a system of forces acting on a rigid body can be reduced to single force R acting at O with a couple of moment G. If X, Y, Z; L, M, N be the six components of the system then

$$X = \sum X_1 = \sum F_1 l_1, Y = \sum Y_1 = \sum F_1 m_1, Z = \sum Z_1 = \sum F_1 n_1 .$$

$$L = \sum (y_1 Z_1 - z_1 Y_1),\ M = \sum (z_1 X_1 - x_1 Z_1),\ N = \sum (x_1 Y_1 - y_1 X_1)$$

with $R^2 = X^2 + Y^2 + Z^2$ and $G^2 = L^2 + M^2 + N^2$.

Let $O'(\xi, \eta, \zeta)$ be other point chosen as base. Let X', Y', Z' ; L', M', N' be corresponding six components of the same system . Since, due to traslation, directions of forces does not change, so

$$X' = \sum X_1' = \sum F_1 l_1 = X, Y' = \sum Y_1' = \sum F_1 m_1 = Y, Z' = \sum Z_1' = \sum F_1 n_1 = Z.$$

Hence resultant w.r.t. base O', is

$$R' = \sqrt{X'^2 + Y'^2 + Z'^2} = \sqrt{X^2 + Y^2 + Z^2} = R$$

i.e., $X^2 + Y^2 + Z^2$ remains unchanged. Therefore for any base point $X^2 + Y^2 + Z^2$ is invariant.

But the couple component depends on the co-ordinates of the point of application of the force. So,

$$L' = \sum \{(y_1 - \eta) Z_1 - (z_1 - \zeta) Y_1\} = \sum (y_1 Z_1 - z_1 Y_1) - \eta \sum Z_1 + \zeta \sum Y_1$$

$$= L - \eta Z + \zeta Y$$

Similarly $M' = M - \zeta X + \xi Z$ and $N' = N - \xi Y + \eta X$

Hence, $L'X' + M'Y' + N'Z' = (L - \eta Z + \zeta Y)X + (M - \zeta X + \xi Z)Y + (N - \xi Y + \eta X)Z$

$$= LX + MY + NZ - \eta XZ + \zeta XY - \zeta XY + \xi YZ - \xi YZ + \eta XZ$$

$$= LX + MY + NZ \cdot$$

Which shows that w.r.t. any base the value of $LX + MY + NZ$ remains unchanged i.e., the quantity $LX + MY + NZ$ is invariant.

Hence the quantities $X^2 + Y^2 + Z^2$ and $LX+MY+NZ$ are invariable.

6.10. Equation of the Central Axis

To find the eqution of the central axis of any given system of forces.

Let a system of forces acting at different points on a rigid body be reduced to a single force R acting at an arbitrary O and a couple of moment G. Let $X, Y, Z; L, M, N$ be the six components of the system such that $R^2 = X^2 + Y^2 + Z^2$, $G^2 = L^2 + M^2 + N^2$ where $X = \sum X_1$, $Y = \sum Y_1$, $Z = \sum Z_1$, $L = \sum (y_1 Z_1 - z_1 Y_1)$, $M = \sum (z_1 X_1 - x_1 Z_1)$, $N = \sum (x_1 Y_1 - y_1 X_1)$ and (x_1, y_1, z_1), (x_2, y_2, z_2) . . . are points of application of system of forces F_1, F_2, . . .having components (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), . . .etc. Let $O'(\xi, \eta, \zeta)$ be any point on the central axis. If the system be now reduced w.r.t. O' as base, then resultant force will be R (invariant) and the couple will be Γ, say, in which the line of action of force R and the axis of the couple Γ will have the same direction. If L', M', N' be the components of Γ where as X, Y, Z remain same, then

$$L' = \sum\{(y_1 - \eta)Z_1 - (z_1 - \zeta)Y_1\} = \sum(y_1 Z_1 - z_1 Y_1) - \eta \sum Z_1 + \zeta \sum Y_1$$

$$= L - \eta Z + \zeta Y$$

$$M' = \sum\{(z_1 - \zeta)X_1 - (x_1 - \xi)Z_1\} = \sum(z_1 X_1 - x_1 Z_1) - \zeta \sum X_1 + \xi \sum Z_1$$

$$= M - \zeta X + \xi Z$$

and $$N' = \sum\{(x_1 - \xi)Y_1 - (y_1 - \eta)X_1\} = \sum(x_1 Y_1 - y_1 X_1) - \xi \sum Y_1 + \eta \sum X_1$$

$$= N - \xi Y + \eta X .$$

Now direction cosines of the line of action R and the axis of couple Γ are $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ and $\frac{L'}{\Gamma}, \frac{M'}{\Gamma}, \frac{N'}{\Gamma}$ respectively. Since the direction of R and the axis of Γ are parallel, so, we have $\frac{L'/\Gamma}{X/R} = \frac{M'/\Gamma}{Y/R} = \frac{N'/\Gamma}{Z/R}$

i.e., $$\frac{L'}{X} = \frac{M'}{Y} = \frac{N'}{Z}$$

or, $$\frac{L - \eta Z + \zeta Y}{X} = \frac{M - \zeta X + \xi Z}{Y} = \frac{N - \xi Y + \eta X}{Z}$$

Hence the locus of the point $O'(\xi, \eta, \zeta)$ is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} \quad \cdots \quad (1)$$

Which is the required equation of the central axis.

Note 1: Also equation of the central axis can be expressed as

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$$

$$= \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2} = \frac{\Gamma}{R} = p$$, p being pitch of the wrench. ... (2)

Note 2 : Equation of the central axis when the wrench of the system reduces to a single force is the intersection of the any two planes.

For a single force, $LX + MY + NZ = 0$. So, from (2) we get $L - yZ + zY = 0$, $M - zX + xZ = 0$, $N - xY + yX = 0$. Solving any two of the above equations, we get the equation of the central axis.