Normal Subgroup

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Normal Subgroup:

Def^n: A subgroup H of a group G is said to be a normal subgroup of G if $aH = Ha$ holds for all $a \in G$.

Note1: The condition $aH = Ha$ does not demand that for every $h \in H$, $ah = ha$.

Note2: When H is a normal subgroup of a group G, there is no distinction between the left cosets and the right cosets of H and we speak simply, 'the cosets of H'.

Otherwise we define normal subgroup as:

A subgroup H of a group G is said to be the normal subgroup of G if every $x \in G$ and for every $h \in H$, $xhx^{-1} \in H$.

Remark: We have $x \in G \Rightarrow x^{-1} \in G$. Therefore H is normal subgroup of G iff $x^{-1}h(x^{-1})^{-1}$ ie $x^{-1}hx \in H$, $\forall x \in G$ and $\forall h \in H$.

Every group possesses at least two normal subgroups.

Note3: The improper subgroup G of a group is a normal subgroup of G.

Proof: Let $a \in G$. Then $aG = G$ and $Ga = G$. Therefore, $aG = Ga$ $\forall a \in G$. This proves that G is normal in G.

Note4: The trivial subgroup $\{e\}$ of a group G is a normal subgroup of G.

Proof: Let $H = \{e\}$ and ~~let~~ $a \in G$. Then $aH = \{a\}$ and $Ha = \{a\}$. Therefore, $aH = Ha$ $\forall a \in G$. This proves that H is normal in G.

Simple group: There exists groups for which these are only normal subgroups. Such groups are called the ~~normal~~ simple group.

Otherwise: A group G is said to be a simple group if G has no normal subgroups other than the trivial and the improper subgroup of G.

Note: Every group of prime order is simple.

Theorem: Let H be a subgroup of a group G and $[G:H]=2$. Then H is normal in G.

Proof: Since $[G:H]=2$, there are exactly two distinct left cosets of H in G. They are H and $G-H$. Also there are two distinct right cosets of H in G. and they are H and $G-H$.

Let $a \in H$. Then $aH = H$ and also $Ha = H$. Clearly $aH = Ha$.

Let $a \in G-H$. Then $aH = G-H$ since $G-H$ is the only left coset other than H. [$\because a \notin H$ then $aH \neq H$. Hence $G = H \cup aH$ and $H \cap aH = \phi$, then $aH = G-H$]

Also, $Ha = G-H$, since $G-H$ is the only right coset other than H. Clearly, $aH = Ha$.

It follows that $aH = Ha \;\forall a \in G$.

Therefore, H is a normal subgroup of G.

Ex: Show that every subgroup of an abelian group is normal subgroup.

Proof: Let G be an abelian group and H be a subgroup of G. Let x be any element of G and h be any element of H.

We have $xhx^{-1} = xx^{-1}h$ [$\because G$ is abelian $\Rightarrow x^{-1}h = hx^{-1}$]

$= e \cdot h$

$= h \in H$.

Thus, $x \in G$, $h \in H \Rightarrow xhx^{-1} \in H$. Hence H is normal in G.

Note: Since every cyclic group is abelian, therefore every subgroup of a cyclic group is normal.

Theorem: Prove that a subgroup H of a group G is normal iff $xHx^{-1} = H \quad \forall x \in G$.

Proof: H is a normal subgroup of $G \Leftrightarrow xH = Hx \quad \forall x \in G$

$\Leftrightarrow xHx^{-1} = Hxx^{-1} \quad \forall x \in G$.

$\Leftrightarrow xHx^{-1} = He = H \quad \forall x \in G$.

Note: A normal subgroup is a itself conjugate of G.

Theorem: Prove that H is a normal subgroup of G iff $xhx^{-1} \in H$ for $h \in H$ and $\forall x \in G$.

Proof: If H is a normal subgroup of G.

Then $xH = Hx \quad \forall x \in G$

$\Rightarrow xHx^{-1} = H \quad \forall x \in G$

$\Rightarrow xhx^{-1} \in H$ ** $\forall h \in H$ and $\forall x \in G$.

** Hence for any $h \in H$, $xh \in xH = Hx$

and so $xh = h'x$ for some $h' \in H$

$\Rightarrow xhx^{-1} = h' \in H$

Conversely Let $xhx^{-1} \in H$ for $h \in H$ and $\forall x \in G$.

$\therefore \Rightarrow xHx^{-1} \subseteq H \quad \forall x \in G$ (1).

Also, $x \in G \Rightarrow x^{-1} \in G$.

We have, $x^{-1} H (x^{-1})^{-1} \subseteq H \quad \forall x \in G$

$\Rightarrow x^{-1} H x \subseteq H \quad \forall x \in G$

$\Rightarrow x(x^{-1} H x)x^{-1} \subseteq xHx^{-1}, \; \forall x \in G$

$\Rightarrow H \subseteq xHx^{-1} \quad \forall x \in G$. —(2)

From (1) and (2), $xHx^{-1} = H \quad \forall x \in G$

$\Rightarrow xH = Hx, \; \forall x \in G$.

It follows that H is a normal subgroup of G.

Theorem: Prove that H is a normal subgroup of G iff the product of two right cosets of H in G is a right coset of H in G.

Proof: Let H be a normal subgroup of G and $a, b \in G$.
Then Ha and Hb are two right cosets of H in G.

Now, $Ha \cdot Hb = H(aH)b$

$= H(Ha)b \quad [\because H \text{ is normal} \Rightarrow Ha = aH]$

$= HHab$

$= Hab \quad [\because HH = H]$

Since $a \in G$, $b \in G \Rightarrow ab \in G$, Hab is a right coset of H in G.

Thus, the product of the cosets Ha and Hb is the coset Hab.

Conversely, Let H be a subgroup of G such that the product of two right cosets of H in G is a right coset of H in G.

Let $x \in G$. Then $x^{-1} \in G$.

Therefore, Hx, Hx^{-1} and $HxHx^{-1}$ are right cosets of H in G.

H is a subgroup, so, $e \in H$.

Therefore $exex^{-1} = e$ and it is an element of $HxHx^{-1}$.

Again He is a right coset and $e \in He$. Since He ie H and $HxHx^{-1}$ have a common element.,

$HxHx^{-1} = H \quad \forall x \in G$

$\Rightarrow h_1 x h x^{-1} \in H, \forall x \in G$ and $\forall h, h_1 \in H$.

$\Rightarrow h_1^{-1}(h_1 x h x^{-1}) \in h_1^{-1} H \quad \forall x \in G$ and $\forall h, h_1 \in H$.

$\Rightarrow x h x^{-1} \in H \quad \forall x \in G$ and $\forall h \in H$

$[\because h_1^{-1} H = H$ as $h_1^{-1} \in H]$

$\Rightarrow H$ is a normal subgroup of G.

Theorem. Prove that intersection of any two normal subgroup of a group is a normal subgroup. V.H. - 2010

Proof: Let H and K be two normal subgroups of a group G. Since H and K are subgroups of G, therefore $H \cap K$ is also a subgroup of G. Now to prove that $H \cap K$ is a normal subgroup of G.

Let x be any element of G and y be any element of $H \cap K$.

We have $y \in H \cap K$

$\Rightarrow y \in H$ and $y \in K$.

Since H is a normal subgroup of G, therefore, $x \in G, y \in H \Rightarrow xyx^{-1} \in H$.

Similarly, $xyx^{-1} \in K$ for K is to be normal subgroup of G.

Now, $xyx^{-1} \in H$, $xyx^{-1} \in K \Rightarrow xyx^{-1} \in H \cap K$.

Hence $H \cap K$ is a normal subgroup of G.

Ex: Prove that a normal subgroup of G is commutative with every non-empty subset of G.

Proof: Let N be a normal subgroup and H be a non-empty subset of the group G.

$n \in N$ and $h \in H \Rightarrow nh \in NH$.

Again $nh = hh^{-1}nh$ $(\because h \in G, h^{-1} \in G)$

$= h(h^{-1}nh)$

Since N is a normal subgroup, $h^{-1}nh \in N$.

Therefore $nh \in HN$ and it implies that $NH \subseteq HN$. (i)

If $h \in H$ and $n \in N \Rightarrow hn \in HN$.

Again, $hn = (hnh^{-1})h \in NH$ $[\because hnh^{-1} \in N]$

$\therefore HN \subseteq NH$ (ii)

$\therefore$ From (i) and (ii), $NH = HN$.

Hence the theorem.

Theorem: If N and M are normal subgroup of G, Prove that NM is also a normal subgroup of G.

Proof: We know that a normal subgroup is commutative with every non-empty ~~set~~ subset of G.

Therefore $NM = MN$.

Now N and M are two subgroups of G such that $NM = MN$.

Therefore NM is a subgroup of G.

Now to show that NM is a normal subgroup of G.

Let x be any element of G and $nm \in NM$. Then $n \in N$ and $m \in M$. We have $x(nm)x^{-1} = (xnx^{-1})(xmx^{-1}) \in NM$

$\left[\begin{array}{l} \because M \text{ is normal} \Rightarrow xmx^{-1} \in M \\ N \text{ is normal} \Rightarrow xnx^{-1} \in N \end{array}\right]$

Therefore, NM is also normal subgroup of G.

Ex: Suppose that N and M are two normal subgroups of G and that $N \cap M = \{e\}$. Show that every element of N commutes with every element of M.

Ans: Let n be any element of N and m be any element of M.

To prove that $nm = mn$.

Consider an element $nmn^{-1}m^{-1}$.

Since N is normal, $mn^{-1}m^{-1} \in N$. Also $n \in N$.

Therefore $nmn^{-1}m^{-1} \in N$ [$\because$ Closure prop]

Again M is normal $\Rightarrow nmn^{-1} \in M$. Also, $m \in M \Rightarrow m^{-1} \in M$.

Therefore $nm\bar{n}^{1}\bar{m}^{1} \in M$.

Thus, $nm\bar{n}^{1}\bar{m}^{1} \in N$ and $nm\bar{n}^{1}\bar{m}^{1} \in M$

$\Rightarrow nm\bar{n}^{1}\bar{m}^{1} \in N \cap M$

$\Rightarrow nm\bar{n}^{1}\bar{m}^{1} = e$ $[\because N \cap M = \{e\}]$

$\Rightarrow nm = mn$.

$\Rightarrow$ Every element of N commutes with every element of M.

Hence the result.

Ex: Let S_n be the symmetric group on n symbols. Prove that A_n is a normal subgroup of S_n.

Ans: Let α be an element of S_n and β be any element of A_n. Then β is an even permutation and α may be even or odd permutation.

We claim that $\alpha\beta\bar{\alpha}^{1}$ is an even permutation.

Case-I: When α = odd permutation.

If α is odd then $\bar{\alpha}^{1}$ is also odd. Now $\alpha\beta$ is odd and consequently $\alpha\beta\bar{\alpha}^{1}$ is even permutation.

Case-II: When α = even permutation,

If α is even then $\bar{\alpha}^{1}$ is even. Now $\alpha\beta$ is even and consequently, $\alpha\beta\bar{\alpha}^{1}$ is even.

Thus, $\alpha \in S_n$, $\beta \in A_n \Rightarrow \alpha\beta\bar{\alpha}^{1} \in A_n$.

Hence A_n is a normal subgroup of S_n.

Ex: Prove that the centre $Z(G)$ of a group G is a normal subgroup of G.
V. H–'2003

Ans: The centre $Z(G) = \{x \in G : xg = gx \ \forall g \in G\}$ is a subgroup of G.

Let $H = Z(G)$. and a be an arbitrary element of G.

Prove that $aH = Ha$ $\forall a \in G$.

Let $p \in aH$: Then $p = ah_1$ for some $h_1 \in H$

$= h_1 a$ since $h_1 \in H = Z(G)$

So, $p \in aH \Rightarrow p \in Ha$ and therefore $aH \subseteq Ha$ —(i)

Let $q \in Ha$. Then $q = h_2 a$ for some $h_2 \in H$

$= ah_2$ since $h_2 \in H = Z(G)$.

So, $q \in Ha \Rightarrow q \in aH$ and therefore, $Ha \subseteq aH$ —(ii).

From (i) and (ii), $aH = Ha$, $\forall a \in G$.

Therefore, H is a normal subgroup of G.

- **Ex**: Prove that alternating group A_3 is a normal subgroup of S_3.

Ans: We know that index of S_3 and A_3 is 2, then A_3 is normal subgroup of S_3.

Here, A_3 is subgroup of S_3. $o(S_3) = 6$, $o(A_3) = 2$.

$$\therefore [S_3 : A_3] = \frac{o(S_3)}{o(A_3)} = 2.$$

Therefore, A_3 is a normal subgroup of S_3.

Note: Otherwise it can be prove by defn

Ex: Prove that $SL(n, R)$ is a normal subgroup of $GL(n, R)$.

Ans: $SL(n,R)$ is the group of all real non-singular $n \times n$ matrices A with $\det A = 1$, and $GL(n,R)$ is the group of all real non-singular $n \times n$ matrices.

Let $G = GL(n, R)$, $H = SL(n, R)$.

$\therefore$ H is a subgroup of G. (Proved previously).

Let $A \in H$, $B \in G$.

Then $\det(BAB^{-1}) = \det B \det A \det B^{-1}$

$= \det(BB^{-1}) \det A$

$= 1$.

It follows that $BAB^{-1} \in H$.

Therefore, $A \in H$, $B \in G \Rightarrow BAB^{-1} \in H$.

This proves that H is a normal subgroup of G.

Ex: Let H be a subgroup of a group G and $a \in G$. Then the subset $aHa^{-1} = \{aha^{-1} : h \in H\}$ is a subgroup of G.

Ans: $e \in H$ and $e = aea^{-1}$ showing that $e \in aHa^{-1}$.

So, aHa^{-1} is non-empty.

Let $p \in aHa^{-1}$ and $q \in aHa^{-1}$.

Then $p = ah_1a^{-1}$, $q = ah_2a^{-1}$ for some $h_1, h_2 \in H$.

$\therefore$ $pq^{-1} = (ah_1a^{-1})(ah_2a^{-1})^{-1}$

$= ah_1a^{-1}(a^{-1})^{-1}h_2^{-1}a^{-1}$

$= ah_1a^{-1}ah_2^{-1}a^{-1}$

$= ah_1h_2^{-1}a^{-1} \in aHa^{-1}$ [H is a subgroup $\Rightarrow$ $h_1 \in H$, $h_2 \in H \Rightarrow h_1h_2^{-1} \in H$.]

Thus, $p \in aHa^{-1}$, $q \in aHa^{-1} \Rightarrow pq^{-1} \in aHa^{-1}$.

It follows that aHa^{-1} is a subgroup of G.

Ex: Let H be a subgroup of a group G. Define $N(H) = \{g \in G : gHg^{-1} = H\}$. Prove that (i) $N(H)$ is a subgroup of G.

(ii) H is normal subgroup $N(H)$.

Ans: (i) $N(H)$ is non-empty since $e \in N(H)$.

Let $p \in N(H)$, $q \in N(H)$. Then $p \cdot H p^{-1} = H$, $qHq^{-1} = H$.

$\therefore (pq) H (pq)^{-1} = p (qHq^{-1}) p^{-1}$

$= pHp^{-1} = H$.

This shows that $pq \in H$. (i)

Let $p \in N(H)$. Then $pHp^{-1} = H$.

$\therefore p^{-1} H (p^{-1})^{-1} = p^{-1} (pHp^{-1}) (p^{-1})^{-1}$

$= eHe^{-1} = H$.

$\Rightarrow p^{-1} \in N(H)$ — (ii)

From (i) and (ii), $N(H)$ is a subgroup of G.

(ii) Let $h \in H$. Then $hHh^{-1} = H$, since H is a subgroup.

This shows that $h \in N(H)$ and therefore, $H \subseteq N(H)$ (i)

$N(H)$ is a subgroup of G, H is a subgroup G and $H \subseteq N(H)$.

Therefore H is a subgroup of $N(H)$.

Let $x \in N(H)$. Then $xHx^{-1} = H$

Since $xHx^{-1} = H$ for every $x \in N(H)$, H is a normal subgroup of $N(H)$.

Hence the result

(Ex): Prove that the set of matrices

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

forms a normal subgroup of the group $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \text{ and } ab \neq 0 \right\}$.

V. H 98

Ans Let $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $A=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$, $B=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$, $C=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

We form a composition table as below

$\cdot$	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

From the composition table we see that S is closed w.r.to matrix multiplication. Matrix multiplication is associative. I is the identity element in S. Inverse of I, A, B, C are respectively I, A, B, C.

Thus S is a commutative group, as each element possesses its own inverse.

Hence every left cosets is equal to its right cosets.

As for example, $\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}-1&0\\0&1\end{pmatrix}=\begin{pmatrix}-a&0\\0&b\end{pmatrix}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}a&0\\0&b\end{pmatrix}$,

Therefore, $\therefore$ S is a normal subgroup of G.

Ex: If H is a subgroup of G such that $x^2 \in H$ $\forall x \in G$. Then prove that H is a normal subgroup of G.

Ans: If $g \in G$ and $h \in H$ consider ghg^{-1} and note that

$$ghg^{-1} = gh\cdot gh\, h^{-1}g^{-2} = (gh)^2 h^{-1} g^{-2}$$

Now, $h^{-1} \in H$ and by our hypothesis, $(gh)^2, g^{-2} \in H$.

$$\therefore \quad ghg^{-1} = (gh)^2 h^{-1} g^{-2} \in H.$$

$\Rightarrow \quad gHg^{-1} \subseteq H.$

$\Rightarrow$ H is a normal subgroup of G.

$\boxed{\text{Theorem}}$: Let H be a subgroup of a group G.

The following conditions are equivalent.

(i) H is a normal subgroup

(ii) $gHg^{-1} \subseteq H$ for all $g \in G$.

(iii) $gHg^{-1} = H$ for all $g \in G$.

Proof: (i) $\Rightarrow$ (ii)
Let H is a normal subgroup of G. Let $g \in G$.

$\therefore$ $gH = Hg$.

Hence for $h \in H$, $gh \in gH = Hg$.

and so $gh = h'g$ for some $h' \in H$.

Thus, $ghg^{-1} = h' \in H$.

Hence $gHg^{-1} \subseteq H$. (ii)

(ii) $\Rightarrow$ (iii). Let $g \in G \Rightarrow g^{-1} \in G$.

Now, $g^{-1}H(g^{-1})^{-1} \subseteq H$

$\Rightarrow g^{-1}Hg \subseteq H$

$\Rightarrow H \subseteq gHg^{-1}$ (2).

Thus, $gHg^{-1} \subseteq H \Rightarrow H \subseteq gHg^{-1}$

$\Rightarrow H = gHg^{-1}$.

(iii) $\Rightarrow$ (i). Since $gHg^{-1} = H$ for all $g \in G$.

$\therefore$ $gH = Hg$ for all $g \in G$.

Hence H is a normal subgroup of G.

(Th): Every subgroup of a commutative group G is a normal subgroup of G.

Proof: Let H be a subgroup of a commutative group G.

Let $a \in G$ Then $aH = \{ah : h \in H\}$ & $Ha = \{ha : h \in H\}$.

Since G is commutative, $ah = ha$ $\forall h \in H$. Therefore

$aH = Ha$ $\forall a \in G$. H is normal in G.