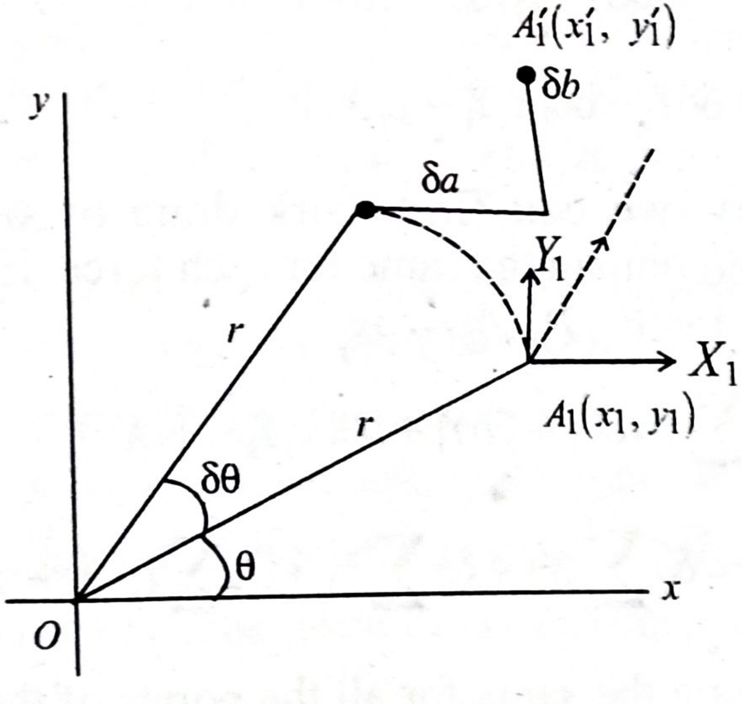
5.2. Principle of Virtual Work for a System of Coplanar Forces

The principle of virtual work states that : *If a system of co-planar forces acting on a rigid body be in equilibrium and the body undergo a slight displacement consistent with the geometrical conditions of the system, the algebraic sum of the virtual works is zero ; and conversely, if this algebraic sum be zero, the forces are in equilibrium.*

Proof. Let a system of co-planar forces $P_1, P_2, \cdots$ be acting at different points $A_1(x_1, y_1), A_2(x_2, y_2), \cdots$ having components $(X_1, Y_1), (X_2, Y_2), \cdots$ parallel to co-ordinate axes Ox and Oy. If (r, θ) be the polar co-ordainates of $A_1(x_1, y_1)$, then $x_1 = r\cos\theta$, $y_1 = r\sin\theta$.



Now, when the body receives a small virtual displacement consistent with the geometrical conditions of the system, the point A_1 takes the new position $A_1'(x_1', y_1')$ by two type of motion : one due to rotation through a small angle $\delta\theta$ and other due to translation through small distances δa and δb parallel to co-ordinate axes.

$$\therefore x_1' = r\cos(\theta+\delta\theta)+\delta a = r\{\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta\}+\delta a$$

$$\text{and } y_1' = r\sin(\theta+\delta\theta)+\delta a = r\{\sin\theta\cos\delta\theta+\cos\theta\sin\delta\theta\}+\delta a$$

$\because$ $\delta\theta$ is very small, so, by first approximation, we have

$$\cos\delta\theta = 1 \text{ and } \sin\delta\theta = \delta\theta$$

$$\therefore x_1' = r\cos\theta - r\delta\theta\cdot\sin\theta+\delta a$$

$$= x_1 - \delta\theta y_1 + \delta a$$

$$\text{and } y_1' = r\sin\theta + r\delta\theta\cos\theta+\delta b$$

$$= y_1 + \delta\theta\cdot x_1 + \delta b.$$

Hence displacement of A_1 along x-axis $= x_1' - x_1$

$$= \delta a - y_1\delta\theta.$$

Similarly displacement of A_1 along y-axis $= y_1' - y_1 = \delta b + x_1\delta\theta.$

Now the virtual work done by the force P_1

= Sum of the works done by its components X_1 and Y_1

= $X_1\cdot$ displacement of A_1 along x-axis $+Y_1\cdot$ displacement of A_1 along y-axis,

$$= X_1(x_1'-x_1)+Y_1(y_1'-y_1)$$

$$= X_1(\delta a - y_1\delta\theta)+Y_1(\delta b + x_1\delta\theta)$$

$$= \delta a X_1 + \delta b Y_1 + \delta\theta(x_1Y_1 - y_1X_1)$$

Similarly, we can find work done by other forces $P_2, P_3, \cdots$ δa, δb and $\delta\theta$ remain the same for each force. Hence total virtual work done by the forces $P_1, P_2 \cdots$ is

$$\sum\{\delta a X_1 + \delta b Y_1 + \delta\theta(x_1Y_1 - y_1X_1)\}$$

$$= \delta a\sum X_1 + \delta b\sum Y_1 + \delta\theta\sum(x_1Y_1 - y_1X_1)\cdots \quad (1)$$

δa, δb, $\delta\theta$ being the same for all the points of the body.

Let the body be in equilibrium under the system of given co-planar forces then,

$$\sum X_1 = 0, \sum Y_1 = 0, \sum(x_1 Y_1 - y_1 X_1) = 0 \quad \ldots \quad (2)$$

Using (2) in (1), we get,

Algebraic sum of the virtual works is zero.

Conversely, let the algebraic sum of the virtual works be zero,

i.e., $$\delta a \sum X_1 + \delta b \sum Y_1 + \delta\theta \sum(x_1 Y_1 - y_1 X_1) = 0 \cdots \quad (3)$$

for all arbitrary small values of δa, δb, and $\delta\theta$. Since the displacement is perfectly arbitrary, δa, δb, $\delta\theta$ are all independent of one another, i.e., the above equations is true for any positive or negative independent values of δa, δb, and $\delta\theta$. Hence we must have

$$\sum X_1 = 0, \sum Y_1 = 0 \text{ and } \sum(x_1 Y_1 - y_1 X_1) = 0.$$

These are the conditions of equilibrium for a system of co-planar forces, therefore the body is in equilibrium.

Note 1. Principle of virtual work is also necessary as well as sufficient conditions.

5.3. Forces which may be omitted in the equation of virtual work

If a body is in equilibrium under the action of a system of forces and if the body is given a small displacement, then the work done by the following forces is zero.

(1) *The tensions of an inextensible string or rod.*

(2) *The reaction of any smooth surface with which the body is in contact.*

Because, the reaction is always normal to the surface and therefore at right angles to the direction of the displacement of its point of action.

(3) *The reaction at any point of contact with a fixed surface on which the body rools without sliding.*

Because, the point of contact is instantaneously at rest, hence the normal reaction and the friction at this point have zero displacement.

(4) *The reactions between any two bodies of the system considered.*

Because, the action and reaction are equal and opposite.

(5) *The reaction at a fixed point or a fixed axis of rotation.*

Because, the displacement of the point of application of the force is zero.

5.4. Forces which may be considered in the equation of virtual work

For virtual displacements which do not violate the geometrical conditions of the system, the following forces may be considered as they contribute to the equation of the virtual work of the system.

(1) *The tensions at the ends of a string when the displacement alters it lengths*. If T be tension and l be the length of the string, then work done is $(-T\delta l)$.

(2) *The thrusts at the ends of a rod when displacement alters the lenghth*. If T be the thrust and l be the length of the rod, then work done is $T\delta l$.

(3) *Weight of a body or weight of a system of bodies.*

If w be the weight and z be the depth of $c.\ g.$ of the body below some fixed plane, then the work done is $w\delta z$ and if z be height of the $c.g.$ above some fixed plane, then the work done is $-w\delta z$. Also if the $c.g.$ does not move vertically upwards or downwards the weight do not do any work.

Note that other than above three forces if there acts any external force then it will be considered in the equation of virtual work.

5.5. An Important Note

If a body or a system of bodies are under the action of no other forces than the weights of the bodies and the natural actions and reactions between them, the positions of equilibrium correspond to the maximum and minimum values of the height of centre of gravity (c.g.) of the system above a fixed horizontal level.

If W be the weight of the system and z the depth / height of c.g. of the system below / above fixed horizontal level and δz the virtual displacement of the c.g. then the equation of virtual work is

$$W\delta z = 0 \text{ or } -W\delta z = 0,$$

since the mutual action and reaction do not appear in the equation of the virtual work. Hence

$$\delta z = 0$$

which $\Rightarrow$ that z is a maximum or minimum.

5.6. Procedure of solving the Problems

(i) Draw the diagram showing the directions of the given forces.

(ii) Replace the string by two forces T and T acting inwards and virtual work done by the tension is $-T\delta l$, where l is the length of the string.

(iii) Replace the rod by two forces T and T acting outwards and virtual work done by the thrust is $T\delta l$, where l is the length of the rod.

(iv) Calculate the distance of the points of applications of various forces from a fixed point or a fixed line. If x be the distance of the point of application of a force F from a fixed point, then the virtual work is $F\delta x$ and this work will be *+ive* or *–ive* according as the distance is measured in the direction of F or in a direction opposite to F.