Real Analysis: (Preliminary Idea of limits)

Limit of a function:

Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of D. A real number l is said to be a limit of f at c if corresponding neighbourhood V of l, there exists a neighbourhood W of c such that $f(x) \in V \ \forall x \in [W - \{c\}] \cap D$.

i.e $\lim_{x \to c} f(x) = l$.

$\epsilon - \delta$ defn:

Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of D. A real number l is said to be a limit if corresponding to a pre-assigned positive ϵ, $\exists$ a $\delta (>0)$ such that $|f(x) - l| < \epsilon \ \forall x \in N'(c, \delta) \cap D$

i.e $|f(x) - l| < \epsilon$ whenever $0 < |x - c| < \delta$

Infinite limits:

Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be the limit point of D. If corresponding to preassigned positive number G, $\exists$ a positive δ such that $f(x) > G \ \forall x \in N'(c, \delta) \cap D$.

i.e $\lim_{x \to c} f(x) = \infty$

Also, Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be the limit point D. If corresponding to a pre-assigned positive number G, $\exists$ a positive δ such that

$f(x) < -G \ \forall x \in N'(c, \delta) \cap D$

i.e $\lim_{x \to c} f(x) = -\infty$.

Limits at infinity:

Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let $(c, \infty) \subseteq D$ for some $c \in \mathbb{R}$. We say f tends to l as $x \to \infty$ if corresponding to preassigned positive ϵ, $\exists$ a real number $G > c$ such that $|f(x) - l| < \epsilon \ \forall x > G$

i.e. $\lim_{x\to\infty} f(x) = l$.

Defn: (Infinite limits at infinity):

Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let $(c,\infty) \subseteq D$ for some $c \in \mathbb{R}$. If corresponding to a pre-assigned positive number G, $\exists$ a real number $K > c$ such that

$$f(x) > G \quad \forall\, x > K$$

i.e. $\lim_{x\to\infty} f(x) = \infty$.

Complex Function (Complex Analysis):

Topology on complex plane: If X is any set then the function $d: X \times X \to \mathbb{R}$ is called a metric or distance function if it satisfies the following conditions for all a, b, c in X

(i) $d(a,b) \geq 0$

(ii) $d(a,b) = 0 \Leftrightarrow a = b$

(iii) $d(a,b) = d(b,a)$

(iv) $d(a,c) \leq d(a,b) + d(b,c)$

(X, d) is called the metric space.

The function $d: \mathbb{C}\times\mathbb{C} \to \mathbb{R}$, $(z, z') \to |z - z'|$ has the following properties

(a) $|z - z'| \geq 0$

(b) $|z - z'| = 0 \Leftrightarrow z = z'$

(c) $|z - z'| = |z' - z|$

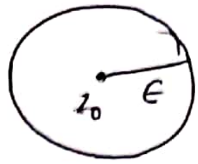
(d) $|z - w| \leq |z - z'| + |z' - w|$ where $z, z', w \in \mathbb{C}$.

Where $d(z,z') = |z - z'|$ is called the Euclidean metric.

Neighbourhood of a point: The neighbourhood of a point z_0 in the complex plane is the set of all points z satisfying $|z - z_0| < \epsilon$, ϵ is small positive constant & is denoted as

$$S(z_0, \epsilon) = \{ z \in \mathbb{C} : |z - z_0| < \epsilon \}.$$

Thus neighbourhood consists of all points of a circular region including the centre z_0 but excluding the points on the boundary of the circle. Such neighbourhood is called the open disk.



Deleted Neighbourhood of z_0:

The deleted neighbourhood of z_0 is the set of points satisfy $|z-z_0|<\epsilon$ excludes the point z_0, denoted as $\delta'(z_0,\epsilon) = \{z \in \mathbb{C} : 0 < |z-z_0| < \epsilon\}$.

Limit point: A point z_0 is a limit point for a set of points in the complex plane if every neighbourhood of z_0 contains points other than z_0 of the given set of points.

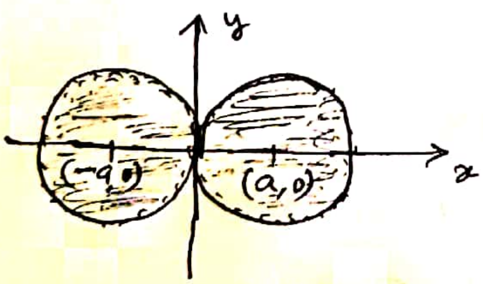
Open set: A subset $S \subseteq \mathbb{C}$ is called open set in $\mathbb{C}$ if every $z_0 \in S$, $\exists\ \delta(>0)$ such that $\Delta(z_0,\delta) \subset S$
i.e some disk arround z_0 lies entirely in S.
Where $\Delta(z_0,\delta)$ is the neighbourhood of z_0.

Disconnected Set: A set $S \subseteq \mathbb{C}$ is said to be seperated or disconnected if there exists two nonempty disjoint open sets A and B such that (i) $S \subseteq A \cup B$,
(ii) $S \cap A \neq \phi$, $S \cap B \neq \phi$

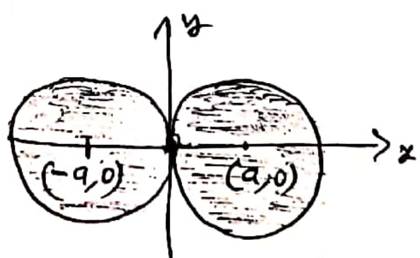
If S is not disconnected, it is called connected.

Ex: If $a>0$, $S_1 = \{z : |z-a| \leq a$ or $|z+a| \leq a\}$
$S_2 = \{z : |z-a| \leq a$ or $|z+a| < a\}$
$S_3 = \{z : |z-a| < a$ or $|z+a| < a\}$

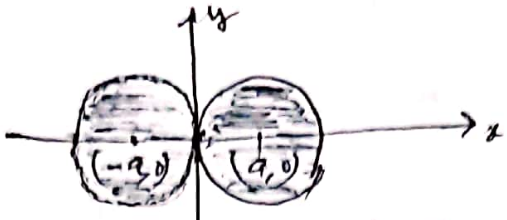
Then S_1 and S_2 are connected whereas S_3 is not connected.



The set by S_3



The set by S_1



The set by S_3

Line segment : Let $z_0, z_1 \in \mathbb{C}$.

The function $\gamma : [0,1] \to \mathbb{C}$ defined by $\gamma(t) = (1-t)z_0 + tz_1$ is called line segment with end points z_0 and z_1, denoted by $[z_0, z_1]$.

Thus $[z_0, z_1] = \{(1-t)z_0 + tz_1 : 0 \le t \le 1\}$

If $\gamma(t) \in S$ for each $t \in [0,1]$, then the line segment $[z_0, z_1]$ is said to be contained in S.

polygonal line : A polygonal line from z_0 to z_n is ~~the~~ a finite union of segments of the form $[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n]$

If $[z_k, z_{k+1}] \subset S$, $k = 0,1,2,\ldots,n-1$, then the polygonal line from z_0 to z_n is said to be contained in S.

Polygonally connected : A set S is said to be polygonally connected if any two points of S can be connected by a polygonal line contained in S.

For example, any open disk $\Delta(z_0, \delta)$ is polygonally connected. For $z_1, z_2 \in \Delta(z_0, \delta)$, $\gamma(t) = (1-t)z_1 + tz_2$, $0 \le t \le 1$.

$$|\gamma(t) - z_0| = |(1-t)(z_1 - z_0) + t(z_2 - z_0)|$$
$$\le (1-t)\delta + t\delta = \delta.$$

$\therefore$ for each $t \in [0,1]$ $\gamma(t) \in \Delta(z_0, \delta)$.

Arcwise Connected Set : A set $S \subseteq \mathbb{C}$ is said to be arcwise connected iff each pair of points $z_0, z_1 \in S$ can be joined by a simple arc lying

~~Wh~~ Wholey with in S.

Every arcwise connected set is connected but the converse may not be true.

Every open set is connected iff it is arcwise connected.

Domain : A domain is a non-empty open connected set in $\mathbb{C}$. A domain togather with some, none or all of its boundary points is referred to as region.

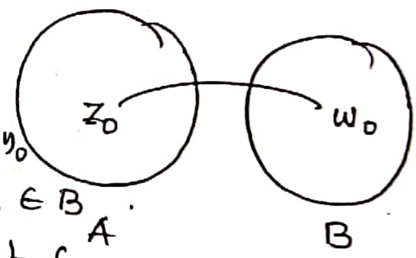
For ex : $S_4 = \{ z \in \mathbb{C} : \text{Re } z < a, a \text{ is real} \}$.

describes a domain.

$S_5 = \{ z \in \mathbb{C} : \text{Re } z \leq a, a \text{ is real} \}$

is a region but is not a domain, since the set S_5 is not open but connected.

Function of a complex variable :



Let A and B be two non empty subsets of $\mathbb{C}$. A function from A to B is a rule f which assigns each $z_0 = x_0 + iy_0 \in A$, a unique element $w_0 = u_0 + iv_0 \in B$. The number w_0 is called the value of f at z_0 & $w_0 = f(z_0)$.

If z varies in A then $w = f(z)$ varies in B. Then f is a complex function of a complex variable in A.

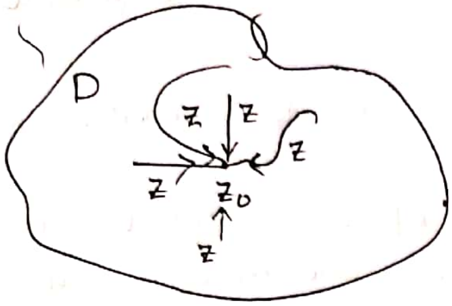
So, $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$.

If w takes only one value for each value of z in A, then w is said to be a single valued function of z.

If there corresponds two or more values of w for some or all values of z in A, $w = f(z)$ is called many valued function or multiple valued function.

Let $f(z) = z^2$ is single valued function but $f(z) = \sqrt{z}$ is a multiple-valued function of z.

Limit of a function: Let f be a complex valued function defined $D \subseteq \mathbb{C}$. Then f is said to have a limit $l (\in \mathbb{C})$ as $z \to z_0$ if for given small positive number ϵ, $\exists$ a positive δ such that



$|f(z) - l| < \epsilon$ whenever $z \in D$ and $0 < |z - z_0| < \delta$.

i.e. for each $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$f(z) \in \Delta(l, \epsilon)$ whenever $z \in [\Delta(z_0, \delta) \setminus \{z_0\}] \cap D$

, $\Delta(z_0, \delta)$ is the n.b.d. of z_0

Limit at infinity: Let f be defined on an unbounded set E. Then for any $R > 0$, $\exists$ $z \in E$ such that $|z| > R$. We say that $f(z) \to l$ as $z \to \infty$ if for every $\epsilon > 0$, $\exists$ an $R > 0$ such that

$|f(z) - l| < \epsilon$ whenever $z \in E$ and $|z| > R$

Ex: Prove that (a) $\lim_{z \to \infty} \frac{1}{z} = 0$ (b) $\lim_{z \to \infty} \frac{1}{z^2} = 0$.

Ans: (a) Let $f(z) = \frac{1}{z}$. $f(z)$ is defined everywhere in $\mathbb{C} \setminus \{0\}$.

Then for every $\epsilon > 0$ $\exists$ $R = \frac{1}{\epsilon}$ such that

$\left|\frac{1}{z}\right| < \epsilon$ whenever $|z| > \frac{1}{\epsilon} = R$.

$\Rightarrow \lim_{z \to \infty} \frac{1}{z} = 0$.

(b). $f(z) = \frac{1}{z^2}$, $f(z)$ is defined everywhere in $\mathbb{C} \setminus \{0\}$.

Then for every $\epsilon > 0$ $\exists$ $R = \frac{1}{\sqrt{\epsilon}}$ s.t.

$\left|\frac{1}{z^2} - 0\right| < \epsilon \Leftrightarrow |z| > \frac{1}{\sqrt{\epsilon}} = R$.

$\therefore$ We choose $R = \frac{1}{\sqrt{\epsilon}}$.

$\therefore \lim_{z \to \infty} \frac{1}{z^2} = 0$.

Ex: Show that $\lim_{z \to 2i} \frac{z^2+4}{z-2i} = 4i$

Ans: Let $f(z) = \frac{z^2+4}{z-2i}$. $f(z)$ is defined $\forall z \in \mathbb{C}$ except $z = 2i$.

Then, for $z \neq 2i$, we have $|f(z) - 4i| = \left|\frac{z^2+4}{z-2i} - 4i\right| = |z - 2i|$.

For any $\epsilon > 0$ $\exists$ $\delta (> 0)$ s.t $|f(z) - 4i| < \epsilon$ whenever $0 < |z - 2i| < \delta$.

so $\delta = \epsilon$.

Hence, $\lim_{z \to 2i} f(z) = \lim_{z \to 2i} \frac{z^2+4}{z-2i} = 4i$.

Defn (Infinite limits): Let $f(z)$ be defined on D except possibly at a point z_0 of D. We say that $f(z) \to \infty$ as $z \to z_0$ if for every $R>0$, $\exists$ a $\delta>0$ such that $|f(z)| > R$ whenever $z \in D \cap N'(z_0, \delta)$.

Q-1: Prove that (i) $\lim\limits_{z\to 1} \frac{1}{|z^2-1|} = \infty$ (ii) $\lim\limits_{z\to 0} \frac{1}{z^2} = \infty$

Ans: (i) Let $f(z) = \frac{1}{z^2-1}$.

The function $f(z)$ is defined $\forall z \in \mathbb{C} - \{1,-1\}$.

Let $R>0$ be given, Then we must show that we can find $\delta>0$ such that $|f(z)| = \left|\frac{1}{z^2-1}\right| > R$ whenever $0<|z-1|<\delta$.

Now, $|f(z)| > R \Leftrightarrow 0 < |z^2-1| < \frac{1}{R}$

Now, $0<|z-1|<\delta \Rightarrow |z^2-1| = |z-1||z-1+2|$

$$\leq |z-1|\{|z-1|+2\}$$
$$\leq \delta(\delta+2) = (\delta+1)^2-1.$$

Therefore, $|z^2-1| < \frac{1}{R}$ if $\delta = \sqrt{1+R^{-1}} - 1$.

Hence $|f(z)| > R$ whenever $z \in \mathbb{C}\setminus\{-1,1\} \cap N'(1, \sqrt{1+R^{-1}}-1)$

$\therefore$ The defn of limit is satisfied.

i.e. $\lim\limits_{z\to 1} \frac{1}{|z^2-1|} = \infty$

(ii) Let $f(z) = \frac{1}{z^2}$ Here, $f(z)$ is defined $\forall z \in \mathbb{C}\setminus\{0\}$.

Let $R(>0)$ be given. We have to show that we can find $\delta>0$ such that $|f(z)| > R$ whenever $0<|z-0|<\delta$.

Now, $|f(z)| = \left|\frac{1}{z^2}\right| = \frac{1}{|z|^2} > R$ whenever $|z| = |z-0| < \delta$

where $\delta = \frac{1}{\sqrt{R}}$

Hence $|f(z)| > R$ whenever $z \in [\mathbb{C}\setminus\{0\}] \cap N'(0, \frac{1}{\sqrt{R}})$.

$\therefore \lim_{z\to 0} \frac{1}{z^2} = \infty$.

Def^n (Infinite limit): Let $f(z)$ be defined on D except possibly at z_0 of D. We say that $f(z) \to \infty$ as $z \to z_0$, for each $\epsilon > 0$, $\exists\ \delta > 0$ such that

$|f(z)| > \frac{1}{\epsilon}$ whenever $z \in D \cap N'(z_0, \delta)$

i.e. $|f(z)| > \frac{1}{\epsilon}$ whenever $0 < |z - z_0| < \delta$. (1).

Here, the statement (1) can be written as $\left|\frac{1}{f(z)} - 0\right| < \epsilon$

whenever $0 < |z - z_0| < \delta$.

Thus $\lim_{z\to z_0} f(z) = \infty$ iff $\lim_{z\to z_0} \frac{1}{f(z)} = 0$.

Ex-2: Prove that $\lim_{z\to 1} \frac{z+3i}{z-1} = \infty$.

Let $f(z) = \frac{z+3i}{z-1}$

Ans: Since $\lim_{z\to 1} \frac{1}{f(z)} = \lim_{z\to 1} \frac{z-1}{z+3i} = 0$.

So $\lim_{z\to 1} f(z) = \infty$.

Def^n: (Infinite limit at infinity):

Let $f(z)$ be defined on an unbounded set E. If for every $R > 0$, $\exists\ K > 0$ such that $|f(z)| > R$ for $|z| > K$ and $z \in E$. Then we say that $f(z) \to \infty$ as $z \to \infty$.

i.e. $\lim_{z\to\infty} f(z) = \infty$.

Otherwise:

Let $f(z)$ be defined on an unbounded set E. If for each $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(z)| > \frac{1}{\epsilon}$ whenever $|z| > \frac{1}{\delta}$.

Replacing z by $\frac{1}{z}$ we have $|f(\frac{1}{z})| > \frac{1}{\epsilon}$ whenever $|\frac{1}{z}| > \frac{1}{\delta}$

$\Rightarrow \left|\frac{1}{f(1/z)} - 0\right| < \epsilon$ whenever $0 < |z| < \delta$

i.e. $0 < |z - 0| < \delta$.

Thus $\lim_{z\to\infty} f(z) = \infty$ iff $\lim_{z\to 0} \frac{1}{f(1/z)} = 0$.

Example: P.T $\lim_{z\to\infty} \frac{z^2+1}{z+1} = \infty$.

Proof: Let $f(z) = \frac{z^2+1}{z+1}$

$$f\left(\frac{1}{z}\right) = \frac{\frac{1}{z^2}+1}{\frac{1}{z}+1}$$

$$\lim_{z\to 0} \frac{1}{f(1/z)} = \lim_{z\to 0} \frac{\frac{1}{z}+1}{\frac{1}{z^2}+1} = \lim_{z\to 0} \frac{z(1+z)}{z^2+1} = 0.$$

Ex-3: Show that when $T(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$

(i) $\lim_{z\to\infty} T(z) = \infty$ if $c=0$

(ii) $\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\lim_{z\to -\frac{d}{c}} T(z) = \infty$ if $c \neq 0$.

Ans: (i) Given that $T(z) = \frac{az+b}{cz+d}$.

$$T\left(\frac{1}{z}\right) = \frac{a/z+b}{c/z+d}.$$

Now, $\lim_{z\to 0} \frac{1}{T(\frac{1}{z})} = \lim_{z\to 0} \frac{c/z+d}{a/z+b}$

$$= \lim_{z\to 0} \frac{c+dz}{a+bz} = \frac{c}{a} = 0 \text{ if } c=0.$$

[$\because a \neq 0$ as $ad-bc \neq 0$]

$\therefore \lim_{z\to\infty} T(z) = \infty$.

(ii) $T(z) = \frac{az+b}{cz+d}$. $\therefore T\left(\frac{1}{z}\right) = \frac{a/z+b}{c/z+d}$.

$$\lim_{z\to 0} T\left(\frac{1}{z}\right) = \lim_{z\to 0} \frac{a/z+b}{c/z+d}$$

$$= \lim_{z\to 0} \frac{a+bz}{c+dz}$$

$$= \frac{a}{c}$$

$\therefore \lim_{z\to 0} T\left(\frac{1}{z}\right) = \frac{a}{c}$ i.e. $\lim_{z\to\infty} T(z) = \frac{a}{c}$ (proved)

Also, $\lim_{z\to -\frac{d}{c}} \frac{1}{T(z)} = \lim_{z\to -d/c} \frac{cz+d}{az+b} = 0$ [$\because ad-bc \neq 0$] $\therefore \lim_{z\to -\frac{d}{c}} T(z) = \infty$ $(c \neq 0)$

Theorem: Prove that limit of a function is unique.

Proof: Let if possible, $\lim_{z\to z_0} f(z) = l_1$, and $\lim_{z\to z_0} f(z) = l_2$, $l_1, l_2 \in \mathbb{C}$.

Then for given $\epsilon > 0$, $\exists$ $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(z) - l_1| < \epsilon/2 \text{ whenever } 0 < |z - z_0| < \delta_1$$

and $$|f(z) - l_2| < \epsilon/2 \text{ whenever } 0 < |z - z_0| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, so $\delta > 0$.

Now, $$|l_2 - l_1| = |(f(z) - l_1) - (f(z) - l_2)|$$
$$\leq |f(z) - l_1| + |f(z) - l_2|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

Since ϵ is any arbitrary small positive number, it follows that $l_2 - l_1 = 0$ so $l_1 = l_2$

Hence the theorem.

Theorem: Let $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$ and $z_0 = x_0 + iy_0$. Let the function f be defined in a domain D except z_0 in D. Then $\lim_{z\to z_0} f(z) = w_0 = u_0 + iv_0$ iff $\lim_{z\to z_0} u(x,y) = u_0$ and $\lim_{z\to z_0} v(x,y) = v_0$.

Proof: We 1st suppose that $\lim_{z\to z_0} f(z)$ exists. Then ~~for~~ every $\epsilon > 0$, $\exists$ $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

i.e. $|u(x,y) + iv(x,y) - (u_0 + iv_0)| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

i.e. $|(u(x,y) - u_0) + i(v(x,y) - v_0)| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

So, $|u(x,y) - u_0| \leq |(u(x,y) - u_0) + i(v(x,y) - v_0)| < \epsilon$ $[\because |\mathrm{Re}\, z| \leq |z|]$

whenever $0 < |z - z_0| < \delta$.

ie $|u(x,y) - u_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

$\Rightarrow \lim_{z\to z_0} u(x,y) = u_0$.

Similarly, $|v(x,y) - v_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$ $(\because |\mathrm{Im}\, z| \leq |z|)$

ie $\lim_{z\to z_0} v(x,y) = v_0$.

Conversely, let $\lim_{z\to z_0} u(x,y) = u_0$ and $\lim_{z\to z_0} v(x,y) = v_0$.

To prove that $\lim_{z\to z_0} f(z) = w_0 = u_0 + i v_0$

Since $\lim_{z\to z_0} u(x,y) = u_0$ and $\lim_{z\to z_0} v(x,y) = v_0$ exists. Then

given $\epsilon > 0$, $\exists$ $\delta_1 > 0$, $\delta_2 > 0$ such that

$|u(x,y) - u_0| < \epsilon/2$ whenever $0 < |z - z_0| < \delta_1$

and $|v(x,y) - v_0| < \epsilon/2$ whenever $0 < |z - z_0| < \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $|f(z) - w_0| = |u(x,y) + i v(x,y) - u_0 - i v_0|$

$\leq |u(x,y) - u_0| + |v(x,y) - v_0|$

$< \epsilon/2 + \epsilon/2 = \epsilon$ whenever $0 < |z - z_0| < \delta$.

$\therefore$ $\lim_{z\to z_0} f(z) = w_0 = u_0 + i v_0$.

$\boxed{\text{Ex}}$. Prove $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

Ans: Let $f(z) = \frac{\bar{z}}{z}$, $z \neq 0$.

$$= \frac{\bar{z}^2}{|z|^2}$$

$$= \frac{(x-iy)^2}{x^2+y^2}, \quad z = x+iy$$

$$= \frac{x^2-y^2}{x^2+y^2} + \frac{-2ixy}{x^2+y^2}$$

$$= u(x,y) + i v(x,y)$$

Where $u(x,y) = \frac{x^2-y^2}{x^2+y^2}$ & $v(x,y) = \frac{-2xy}{x^2+y^2}$.

Let us approach $z \to 0$ along line $y = mx$, m being real no.

We have $f(x+imx) = \frac{1-m^2}{1+m^2} - \frac{2mi}{1+m^2}$, which has different value for different values of m.

$\therefore$ $\lim_{z\to 0} f(z)$ does not exist.

Theorem: If $\lim_{z\to a} f(z)=p$, $\lim_{z\to a}\phi(z)=q$ where $z=x+iy$, $f(z)=u(x,y)$ $+iv(x,y)$, $p=p_0+iq_0$, $q=r_0+is_0$, $a=a_0+ib_0$ Then

(i) $\lim_{z\to a}[f(z)\pm\phi(z)]=p\pm q$

(ii) $\lim_{z\to a}[f(z)\,\phi(z)]=pq$

(iii) $\lim_{z\to a}\frac{f(z)}{\phi(z)}=\frac{p}{q}$, $q\neq 0$.

(iv) $\lim_{z\to a}|f(z)|=|p|$

(v) $\lim_{z\to a} z^n = a^n$.

(vi) $\lim_{z\to a} p(z)=p(a)$ where $p(z)$ is a polynomial in z of the form $a_0+a_1z+\cdots+a_nz^n$, $a_n\neq 0$.

Proof: (i) Since $\lim_{z\to a} f(z)=p$ and $\lim_{z\to a}\phi(z)=q$.

Then for given $\epsilon_1,\epsilon_2>0$ $\exists$ $\delta_1>0$ & $\delta_2>0$ such that

$|f(z)-p|<\epsilon_1$ whenever $0<|z-a|<\delta_1$

& $|\phi(z)-q|<\epsilon_2$ whenever $0<|z-a|<\delta_2$

Now $\delta=\min\{\delta_1,\delta_2\}$.

Then for all z with $0<|z-a|<\delta$, it follows from triangle inequality that $|f(z)+\phi(z)-(p+q)|\leq|f(z)-p|+|\phi(z)-q|<(\epsilon_1+\epsilon_2)$

$\because$ ϵ_1,ϵ_2 are arbitrary.

$\therefore$ $\lim_{z\to a}[f(z)+\phi(z)]=p+q$.

Similarly, $\lim_{z\to a}[f(z)-\phi(z)]=p-q$.

(ii) Now for all z satisfying $0<|z-a|<\delta$, it follows from the triangle inequality that

$$|f(z)\phi(z)-pq|=|\phi(z)(f(z)-p)+p(\phi(z)-q)|$$
$$\leq|\phi(z)|\,|f(z)-p|+|p|\,|\phi(z)-q|$$
$$\leq[|\phi(z)-q+q|\,|f(z)-p|+|p|\,|\phi(z)-q|]$$
$$\leq[|\phi(z)-q|+|q|]\,|f(z)-p|+|p|\,|\phi(z)-q|$$
$$\leq[\epsilon_2+|q|]\,\epsilon_1+|p|\,\epsilon_2.$$

Since ϵ_1 and ϵ_2 are arbitrary,

$\therefore$ $\lim_{z\to a} f(z)\,\phi(z)=pq$

(iii) If we choose $\epsilon_2 = \frac{|q|}{2}$ and $-|\varphi(z)| + q \leq |\varphi(z) - q|$.

Then $|\varphi(z) - q| < \epsilon_2$ whenever $0 < |z-a| < \delta_2$.

i.e. $|\varphi(z)| > \frac{|q|}{2}$ whenever $0 < |z-a| < \delta_2$.

So, $\varphi(z) \neq 0$ in the deleted neighbourhood $\Delta(a, \delta_2) \setminus \{a\}$. Now for all z satisfying $0 < |z-a| < \delta$, it follows that for $\epsilon_2 \leq \frac{|q|}{2}$;

$$\left| \frac{f(z)}{\varphi(z)} - \frac{p}{q} \right| = \left| \frac{f(z)\,q - p\,\varphi(z)}{\varphi(z)\,q} \right|$$

$$= \left| \frac{(f(z)-p)\,q - p(\varphi(z) - q)}{\varphi(z)\,q} \right|$$

$$\leq \left[\epsilon_1 |q| + \epsilon_2 |p| \right] \left[\frac{2}{|q|} \cdot \frac{1}{|q|} \right]$$

$$= \frac{2(\epsilon_1 |q| + \epsilon_2 |p|)}{|q|^2}$$

Since ϵ_1 and ϵ_2 are arbitrary.

$\therefore \left| \frac{f(z)}{\varphi(z)} - \frac{p}{q} \right| < \epsilon$ whenever $0 < |z-a| < \delta$.

$\Rightarrow \lim_{z \to a} \frac{f(z)}{\varphi(z)} = \frac{p}{q}$, $q \neq 0$.

(iv) For given $\epsilon > 0$ $\exists$ a positive δ such that

$||f(z)| - |p|| < |f(z) - p| < \epsilon$ whenever $0 < |z-a| < \delta$.

[$\because$ Since $\lim_{z \to a} f(z) = p$]

$\therefore \lim_{z \to a} |f(z)| = |p|$.

(v) For $n=1$, $|z-a| < \epsilon$ whenever $0 < |z-a| < \delta (= \epsilon)$ [$\because$ For every $\epsilon > 0$, $\exists \delta$]

$\therefore \lim_{z \to a} z = a$

Hence, the result is true for $n=1$.

Assume that the result is true for $n=m$, m being positive integer.

Then $\lim_{z \to a} z^m = a^m$.

Now, $\lim_{z\to a} z^{m+1} = \lim_{z\to a} z\cdot z^m = \lim_{z\to a} z \lim_{z\to a} z^m = a\cdot a^m = a^{m+1}$.

By mathematical induction, the result is true for $n=m+1$ if it is true for $n=m$. Hence, the result is true for any positive integer n.

$\therefore \lim_{z\to a} z^n = a^n$.

(vi)

$\lim_{z\to a} p(z)$

$= \lim_{z\to a} [a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n]$

$= \lim_{z\to a} a_0 + a_1 \lim_{z\to a} z + \cdots + a_n \lim_{z\to a} z^n$ [$\because$ By (i)]

$= a_0 + a_1 a + a_2 a^2 + \cdots + a_n a^n$ [By (v)]

$= p(a)$.

$\boxed{\text{Ex}}$: Show that $\lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2$ does not exist.

Ans:

Let $z = x+iy$, $\bar{z} = x-iy$.

$\therefore \lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2 = \lim_{(x,y)\to(0,0)} \frac{(x+iy)^2}{(x-iy)^2}$

Let us approach origin $(0,0)$ along the line $y=mx$, we have,

$\lim_{(x,y)\to(0,0)} \left(\frac{x+iy}{x-iy}\right)^2 = \frac{(1+im)^2}{(1-im)^2} = \frac{(1+im)^4}{(1+m^2)^2}$

which is different for different values of m.

Therefore, $\lim_{(x,y)\to(0,0)} \frac{(x+iy)^2}{(x-iy)^2}$ does not exist.

Hence $\lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2$ does not exist.

$\boxed{\text{Ex}}$: (i) Find $\lim_{z\to i} \frac{iz^3-1}{z+i}$

(ii) Let $\Delta z = z - z_0$ Then show that $\lim_{z\to z_0} f(z) = w_0$ iff $\lim_{\Delta z\to 0} f(z_0+\Delta z) = w_0$.

(iii) Show that $\lim_{z\to z_0} f(z)g(z) = 0$ if $\lim_{z\to z_0} g(z) = 0$ and if $\exists$ a positive M such that $|f(z)| \le M$ $\forall z$ in the n.b.d. of z_0.

Ans: (i) $\lim_{z\to i} \frac{iz^3-1}{z+i} = \lim_{z\to i} \frac{iz^3 - i^4}{z+i} = \lim_{z\to i} \frac{i(z^3-i^3)}{z+i} = \frac{0}{2i} = 0$,

(ii) $\Delta z = z - z_0$, As $z\to z_0$; $\Delta z \to 0$

and $z = z_0 + \Delta z$.

Then $\lim_{z \to z_0} f(z) = w_0$

$\Rightarrow \lim_{\Delta z \to 0} f(z_0 + \Delta z) = w_0$

Conversely, $\lim_{\Delta z \to 0} f(z_0 + \Delta z) = w_0$

Put $z_0 + \Delta z = z$ Then $\Delta z = z - z_0$. As $\Delta z \to 0$, $z \to z_0$.

$\therefore \lim_{\Delta z \to 0} f(z_0 + \Delta z) = w_0$

$\Rightarrow \lim_{z \to z_0} f(z) = w_0$

(iii) $\lim_{z \to z_0} g(z) = 0$. Then for given $\epsilon > 0$, $\exists$ a positive δ and s.t. $|g(z)| < \frac{\epsilon}{M}$ whenever $0 < |z - z_0| < \delta$.

where M is a positive real number.

Now, $|f(z)\, g(z) - 0| = |f(z)|\,|g(z)| \leq M\,|g(z)|$ [Since $|f(z)| \leq M$ $\forall z$ in the n.b.d. of z_0]

$< M \cdot \frac{\epsilon}{M} = \epsilon$

whenever $0 < |z - z_0| < \delta$.

$\therefore \lim_{z \to z_0} f(z)\, g(z) = 0$.

Continuity of a function $f(z)$:

Def^n: Let f be function of complex variable z and let it be defined in some n.b.d of a point z_0. Then f is said to be continuous at a point z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

ϵ–δ def^n: The function $f(z)$ is said to be continuous at z_0 if given any positive ϵ, we can find a positive number δ s.t $|f(z) - f(z_0)| < \epsilon$ provided $|z - z_0| < \delta$.

Theorem: Let $f(z)$ be defined in some neighbourhood of the point $z_0 = x_0 + iy_0$. Let $f(z) = u(x,y) + iv(x,y)$. Then f is continuous at z_0 iff both $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0).

Proof: Let f be continuous at z_0.
Then corresponding to $\epsilon > 0$, we can find a positive

number δ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever } |z - z_0| < \delta.$$

i.e $\left| u(x,y) + iv(x,y) - \{u(x_0,y_0) + iv(x_0,y_0)\} \right| < \epsilon$

whenever $(x-x_0)^2 + (y-y_0)^2 < \delta^2$.

$\Rightarrow$ $\left| [u(x,y) - u(x_0,y_0)] + i[v(x,y) - v(x_0,y_0)] \right| < \epsilon$

whenever $(x-x_0)^2 + (y-y_0)^2 < \delta^2$

This implies that $|u(x,y) - u(x_0,y_0)| < \epsilon$ and $|v(x,y) - v(x_0,y_0)| < \epsilon$

whenever $(x-x_0)^2 + (y-y_0)^2 < \delta^2$.

This shows that $u(x,y)$ is continuous at (x_0,y_0) and $v(x,y)$ is also continuous at (x_0,y_0).

Conversely, let both $u(x,y)$ & $v(x,y)$ are continuous at (x_0,y_0). We choose $\epsilon > 0$ arbitrarily, Then we can find positive numbers δ_1 and δ_2 such that

$|u(x,y) - u(x_0,y_0)| < \epsilon/2$ ~~and~~ whenever $(x-x_0)^2 + (y-y_0)^2 < \delta_1^2$

and $|v(x,y) - v(x_0,y_0)| < \epsilon/2$ whenever $(x-x_0)^2 + (y-y_0)^2 < \delta_2^2$

Then $|f(z) - f(z_0)|$

$$= \left| \{u(x,y) + iv(x,y)\} - \{u(x_0,y_0) + iv(x_0,y_0)\} \right|$$

$$\leq |u(x,y) - u(x_0,y_0)| + |v(x,y) - v(x_0,y_0)|$$

$< \epsilon/2 + \epsilon/2 = \epsilon$ whenever $(x-x_0)^2 + (y-y_0)^2 < \min\{\delta_1^2, \delta_2^2\}$.

ie $|z - z_0| < \delta$, $\delta = \min\{\delta_1, \delta_2\}$.

$\therefore$ $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

$f(z)$ is continuous at z_0.

$\boxed{\text{Ex}}$: Let $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$

$= 0$, $z = 0$.

Prove that f is continuous on D.

Ans: Here $f(z) = \dfrac{x^3 - y^3}{x^2+y^2} + i\,\dfrac{(x^3+y^3)}{x^2+y^2}$

$= u(x,y) + i v(x,y)$ where $u(x,y) = \frac{x^3-y^3}{x^2+y^2}$, $v(x,y) = \frac{x^3+y^3}{x^2+y^2}$.

When $z \neq 0$, $x^2+y^2 \neq 0$.

so $u(x,y)$, $v(x,y)$ are two rational function of x and y with non zero denominator. Also x^3-y^3, x^3+y^3 being polynomial in x & y are continuous.

Therefore, for $z \neq 0$, $u(x,y)$ and $v(x,y)$ are continuous.

Now for $z=0$, we approach $z \to 0$ along the line $y = mx$ we have.

$$\lim_{(x,y)\to(0,0)} u(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0} \frac{x^3(1-m^3)}{x^2(1+m^2)} = 0 = f(0).$$

and $$\lim_{(x,y)\to(0,0)} v(x,y) = \lim_{x\to 0} \frac{x^3(1+m^3)}{x^2(1+m^2)} = 0 = f(0)$$

So $u(x,y)$ and $v(x,y)$ are also continuous at $(0,0)$.

Hence, $f(z)$ is continuous $\forall z \in D$.

$\boxed{\text{Ex}}$: Show that $f(z) = |z|^2$ is continuous for all $z \in \mathbb{C}$.

Ans: Let $z = x+iy$.

$\therefore f(z) = x^2+y^2$

so $u(x,y) = x^2+y^2$ & $v(x,y) = 0$.

Since $u(x,y)$ and $v(x,y)$ are continuous functions of x and y for all $(x,y) \in \mathbb{R}^2$. So, $f(z)$ is continuous $\forall z \in \mathbb{C}$.

$\boxed{\text{Ex}}$ If $f(z)$ is continuous at $z = z_0$ then show that $\bar{f}(z)$ is also continuous at $z = z_0$.

Ans: Since $f(z)$ is continuous at $z = z_0$. Then for given $\epsilon > 0$, $\exists$ a positive δ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Now, $|\bar{f}(z) - \bar{f}(z_0)| = |\overline{f(z) - f(z_0)}| = |f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

$\therefore$ This implies that $\bar{f}(z)$ is continuous at $z = z_0$.

$\boxed{\text{Ex}}$ If $f(z)$ be continuous at $z=a$ and $f(a) \neq 0$ then $f(z) \neq 0$ $\forall z$ in the neighbourhood $|z-a| < \delta$.

Ans: Since $f(z)$ is continuous at $z=a$, for every $\epsilon > 0$, $\exists$ a number $\delta (>0)$ s.t $|f(z) - f(a)| < \epsilon$ whenever $|z-a| < \delta$

We choose $\epsilon = \frac{|f(a)|}{2}$ $\quad [\because f(a) \neq 0]$

Then $|f(z) - f(a)| < \frac{|f(a)|}{2}$ in the nbd $|z-a|<\delta$ Then

Now, if $f(z)=0$ for some point z in the nbd $|z-a|<\delta$ then

$|f(a)| < \frac{|f(a)|}{2}$ which is impossible.

Hence $f(z) \neq 0 \; \forall z$ in the nbd $|z-a|<\delta$.

$\boxed{\text{Theorem}}$: If $f(z) = u(x,y) + i v(x,y)$ be continuous on D then f is bounded on D. That is if f is continuous on D, then there exist a positive real number M such that $|f(z)| \leq M \; \forall z \in D$.

Proof : Given $f(z) = u(x,y) + i v(x,y)$

$$|f(z)| = \sqrt{u^2 + v^2}$$

Since $f(z)$ is continuous on D, then $u(x,y)$ & $v(x,y)$ are continuous on D. So, obviously $u(x,y)$ and $v(x,y)$ are bounded on D. So $|f(z)|$ is bounded on D. Then $\exists$ a positive real number M such that $|f(z)| \leq M \; \forall z \in D$.

$\boxed{\text{Ex}}$: Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is continuous everywhere.

Ans: Given $f(z) = \sin x \cosh y + i \cos x \sinh y$

$= u(x,y) + i v(x,y)$

$\therefore u(x,y) = \sin x \cosh y$ and $v(x,y) = \cos x \sinh y$.

$u(x,y)$ and $v(x,y)$ are trigonometric functions of x and y.

So $u(x,y)$ and $v(x,y)$ are continuous $\forall (x,y) \in \mathbb{R}^2$.

$\therefore f(z)$ is continuous everywhere.

Differentiability :

Let f be a complex function defined in a non-empty open set D.

The function $f(z)$ is differentiable at $z_0 \in D$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

i.e. $\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ exists & denoted by $f'(z_0)$.

A function which is differentiable in a entire complex plane is called an entire function.