Vector Application to Mechanics

Semester-II
Paper- C201

Course: Mathematics (H)

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Vectors Application to Mechanics :

Theorem of Rankine : (Four forces in equilibrium)

If four forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, \vec{P_4}$ acting at a point are in equilibrium then each force is proportional to the volume of the parallelopiped determined by the unit vectors in the directions of other three.

Proof : Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the unit vectors in the direction of forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, \vec{P_4}$ and their magnitudes are $|\vec{P_1}|, |\vec{P_2}|, |\vec{P_3}|, |\vec{P_4}|$ respectively so that

$$\vec{P_1} = |\vec{P_1}|\,\vec{a}$$
$$\vec{P_2} = |\vec{P_2}|\,\vec{b}$$
$$\vec{P_3} = |\vec{P_3}|\,\vec{c}$$
and
$$\vec{P_4} = |\vec{P_4}|\,\vec{d}$$

Since the forces are in equilibrium, their vector sum is zero.

Thus, $\vec{P_1} + \vec{P_2} + \vec{P_3} + \vec{P_4} = 0$

$$\Rightarrow |\vec{P_1}|\,\vec{a} + |\vec{P_2}|\,\vec{b} + |\vec{P_3}|\,\vec{c} + |\vec{P_4}|\,\vec{d} = 0 \quad (1)$$

Multiplying (1) scalarly by $\vec{a}\times\vec{b}$, $\vec{a}\times\vec{c}$ and $\vec{c}\times\vec{d}$ we have

$$\left.\begin{aligned} |\vec{P_3}|\,[\vec{c}\,\vec{a}\,\vec{b}] + |\vec{P_4}|\,[\vec{d}\,\vec{a}\,\vec{b}] &= 0 \\ |\vec{P_2}|\,[\vec{b}\,\vec{a}\,\vec{c}] + |\vec{P_4}|\,[\vec{d}\,\vec{a}\,\vec{c}] &= 0 \\ |\vec{P_1}|\,[\vec{a}\,\vec{c}\,\vec{d}] + |\vec{P_2}|\,[\vec{b}\,\vec{c}\,\vec{d}] &= 0. \end{aligned}\right\} (2)$$

With the proper choice of signs, we have from (2)

$$\frac{|\vec{P_1}|}{[\vec{b}\,\vec{c}\,\vec{d}]} = \frac{|\vec{P_2}|}{[\vec{c}\,\vec{a}\,\vec{d}]} = \frac{|\vec{P_3}|}{[\vec{a}\,\vec{b}\,\vec{d}]} = \frac{|\vec{P_4}|}{[\vec{b}\,\vec{a}\,\vec{c}]}$$

Thus, each force is proportional to the scalar triple product of unit vectors in the directions of the other three forces and hence to the volume of the parallelopiped determine by unit vectors.

Lami's theorem: (Three forces in equilibrium)

Statement: If three concurrent forces be in equilibrium, then they are coplanar and each is proportional to the sine of the angle between other two.

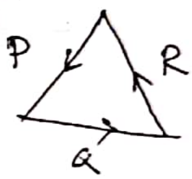
Proof: Let $\vec{P}, \vec{Q}, \vec{R}$ be the three concurrent forces and P, Q, R be the magnitudes of the three concurrent forces.

Let $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be the unit vectors in the direction of forces $\vec{P}, \vec{Q}, \vec{R}$ respectively.

Here, the forces acting at a point are in equilibrium. So, they form a closed polygon. Here, the polygon is triangle. Hence, the vectors are coplanar.

Since the forces are in equilibrium, we have

$$P\vec{\alpha} + Q\vec{\beta} + R\vec{\gamma} = \vec{0} \quad \text{---(1)}$$



Multiplying vectorially (1) by $\vec{\alpha}$ we have

$$P(\vec{\alpha} \times \vec{\alpha}) + Q\,\vec{\beta} \times \vec{\alpha} + R\,\vec{\gamma} \times \vec{\alpha} = \vec{0}$$

$$\Rightarrow Q(\vec{\alpha} \times \vec{\beta}) = R(\vec{\gamma} \times \vec{\alpha}) \quad \text{---(2)}$$

$$\Rightarrow \frac{Q}{|\vec{\gamma} \times \vec{\alpha}|} = \frac{R}{|\vec{\alpha} \times \vec{\beta}|}$$

Similarly, multiplying vectorially (1) by $\vec{\beta}$, we have

$$P(\vec{\alpha} \times \vec{\beta}) + R(\vec{\gamma} \times \vec{\beta}) = 0$$

$$\Rightarrow \frac{P}{|\vec{\beta} \times \vec{\gamma}|} = \frac{R}{|\vec{\alpha} \times \vec{\beta}|} \quad \cdots\cdots (3).$$

From (2) and (3) we have

$$\frac{P}{|\vec{\beta} \times \vec{\gamma}|} = \frac{Q}{|\vec{\gamma} \times \vec{\alpha}|} = \frac{R}{|\vec{\alpha} \times \vec{\beta}|}.$$

$$\Rightarrow \frac{P}{\sin \widehat{\beta\gamma}} = \frac{Q}{\sin \widehat{\gamma\alpha}} = \frac{R}{\sin \widehat{\alpha\beta}}$$

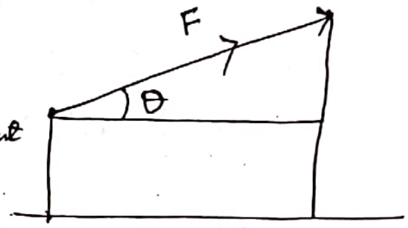
Since $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are unit vectors.

Article : Work done by a force

A force acting on a particle, does work when the particle is displaced in a direction which is not perpendicular to the force.

So, the work done by the force acting on a particle is a scalar quantity, measured by the product of magnitude of the force and the resolved part of the displacement in the direction of the force.

If $\vec{F}$ and $\vec{d}$ are the vectors representing the force and displacement inclined at an angle θ, then



$$W = F d\cos\theta = |\vec{F}|\,|\vec{d}|\cos\theta = F\cos\theta \times d$$
$$= \vec{F}\cdot\vec{d} \ldots$$

$\boxed{\text{Ex-1}}$: A particle being acted by constant forces $(4\hat{i}+\hat{j}-3\hat{k})$ and $(3\hat{i}+\hat{j}-\hat{k})$ is displaced from the point $(\hat{i}+2\hat{j}+3\hat{k})$ to the point $(5\hat{i}+4\hat{j}-\hat{k})$. Find the total work done by the forces.

[V.H-87, 2000]

Ans : Let $\vec{F_1} = 4\hat{i}+\hat{j}-3\hat{k}$ and $\vec{F_2} = 3\hat{i}+\hat{j}-\hat{k}$.

We know that the work done by several forces is equal to the work done by resultant force.

$\therefore$ Resultant force $\vec{F} = \vec{F_1}+\vec{F_2}$

$$= (4\hat{i}+\hat{j}-3\hat{k}) + (3\hat{i}+\hat{j}-\hat{k})$$
$$= 7\hat{i}+2\hat{j}-4\hat{k}.$$

The point is displaced from $P(\hat{i}+2\hat{j}+3\hat{k})$ to $Q(5\hat{i}+4\hat{j}-\hat{k})$.

So the displacement is $\vec{d} = \vec{PQ} = \vec{OQ} - \vec{OP}$

$= $ Position vector of Q − Position vector of P

$= (5\hat{i} + 4\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$.

$= 4\hat{i} + 2\hat{j} - 4\hat{k}$

∴ Total work done $= \vec{F} \cdot \vec{d}$

$= (7\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (4\hat{i} + 2\hat{j} - 4\hat{k})$

$= 28 + 4 + 16$

$= 48$ units

$\boxed{\text{Ex-2}}$: ~~A force of 15 units acts in the direction~~

Forces acting on a particle having magnitude 5, 3, 1 lbs weight and act in the direction of vector $6\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} - 2\hat{j} + 6\hat{k}$ and $2\hat{i} - 3\hat{j} - 6\hat{k}$ respectively. These remain constant when the particle is displaced from the point $A(2\hat{i} - \hat{j} + 3\hat{k})$ to $B(5\hat{i} - \hat{j} + \hat{k})$.

Ans: Let $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_2}$ be the forces in the direction $6\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} - 2\hat{j} + 6\hat{k}$ and $2\hat{i} - 3\hat{j} - 6\hat{k}$ resp. of magnitude 5, 3, 1 resp. ∴

$$\vec{F_1} = 5\frac{(6\hat{i} + 2\hat{j} + 3\hat{k})}{7}.$$

$$\vec{F_2} = 3\frac{(3\hat{i} - 2\hat{j} + 6\hat{k})}{7}.$$

$$\vec{F_3} = \frac{(2\hat{i} - 3\hat{j} - 6\hat{k})}{7}.$$

Let $\vec{F}$ be the resultant force.

$$\therefore \vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3}$$

$$= \frac{1}{7}(41\hat{i} + \hat{j} + 27\hat{k}).$$

The displacement $\vec{d} = \vec{AB} = \vec{OB} - \vec{OA}$

$= (5\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k})$

$= 3\hat{i} - 2\hat{k}$.

We know that the work done by system of forces acting on a particle is equal to the work done by their resultant.

$\therefore$ Total work done $= \vec{F} \cdot \vec{d}$

$$= \frac{1}{7}(41\hat{i} + \hat{j} + 27\hat{k}) \cdot (3\hat{i} - 2\hat{k})$$

$$= \frac{1}{7}(123 - 54) \text{ units}$$

$$= \frac{69}{7} \text{ units.}$$

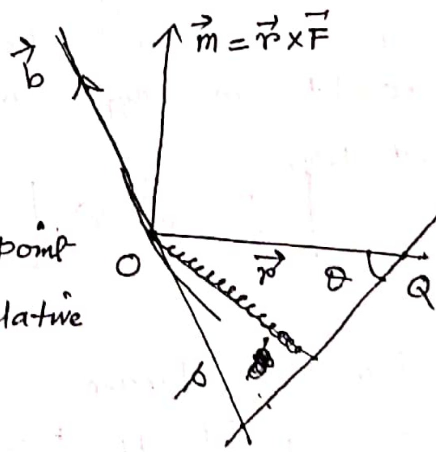
$\boxed{\text{Ex-3}}$: A particle acted on by constant forces $(5\hat{i} + 2\hat{j} + \hat{k})$ and $(2\hat{i} - \hat{j} - 3\hat{k})$ is displaced from the origin to the point $(\hat{i} + \hat{j} - 3\hat{k})$, show that the total work done by the forces is 35 units of work.

Ans: Try yourself.

$\boxed{\text{Ex-4}}$: Forces 6, 7, 2 pounds weight act on a particle along the vectors $(6,2,3)$, $(3,-2,6)$ and $(2,-3,-6)$ respectively. If the particle be displaced from the point $(2,-1,-3)$ to the point $(5,-1,1)$ then show that the work done $53\frac{4}{7}$ foot-pounds weight, the unit of length being one foot. [Ans: $\frac{375}{7}$ units]

Article: Moment of a force about a point

The moment (or torque) of a force $\vec{F}$ about the point O is the vector $\vec{m} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the position vector of any point Q on the line of action of $\vec{F}$ relative to O.



Note: (i) We define the projection of $\vec{m}$ on $\vec{b}$ as the moment of the force $\vec{F}$ about the line $\vec{b}$.

Thus, the moment of $\vec{F}$ about the line $\vec{b}$ is given by

$$\vec{m}\cdot\vec{b} = (\vec{r}\times\vec{F})\cdot\vec{b}.$$

(ii) Magnitude of $\vec{m}$ is $|\vec{r}\times\vec{F}|$

$$= |\vec{r}|\,|\vec{F}|\sin\theta \quad \left[\because \theta \text{ is the angle between } \vec{F} \text{ and } \vec{r}\right]$$

$$= |\vec{F}|\,|\vec{r}|\sin\theta$$

$$= |\vec{F}|\,p \quad \left[\because p = |\vec{r}|\sin\theta\right]$$

= ⊥r distance from O on the line of action of force $\vec{F}$.

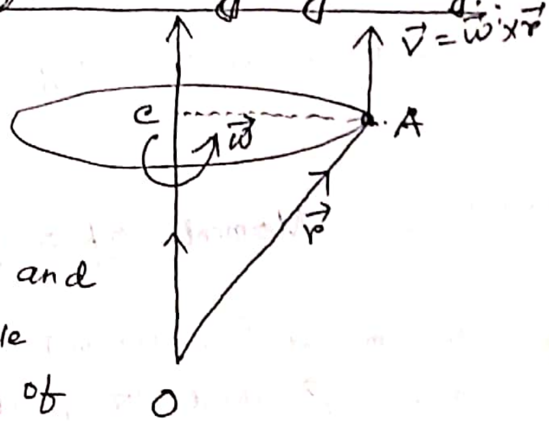
(iii) If the number of forces $\vec{F_1}, \vec{F_2}, \dots, \vec{F_n}$ passing through Q, we have the sum of moments

$$= \vec{r}\times\vec{F_1} + \vec{r}\times\vec{F_2} + \dots + \vec{r}\times\vec{F_n}$$

$$= \vec{r}\times(\vec{F_1} + \vec{F_2} + \dots + \vec{F_n})$$

$$= \vec{r}\times\vec{F} \text{ where } \vec{F} = \sum_{i=1}^{n}\vec{F_i}.$$

Article: <u>Velocity at a point of a rotating rigid body.</u>



The velocity of any particle of a body revolving about the fixed axis is equal to the vector product of the angular velocity and the position vector of the particle referred to an origin on the axis of rotation.

$$\therefore \boxed{\vec{V} = \vec{\omega}\times\vec{r}.}$$

Ex-5: Find the torque about the point B(3,−1,3) of a force P(4,2,1) passing through the point A(5,2,4).

Ans: The position vector of A relative to B is

$$\vec{r} = \vec{OA} - \vec{OB} = (5\hat{i}+2\hat{j}+4\hat{k}) - (3\hat{i}-\hat{j}+3\hat{k})$$

$$= 2\hat{i}+3\hat{j}+\hat{k}.$$

The force $\vec{F}$ is $4\hat{i}+2\hat{j}+\hat{k}$.

Hence, the required moment is $\vec{r}\times\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}+2\hat{j}-8\hat{k}.$$

$\boxed{\text{Ex-6}}$: A force of 15 units acts in the direction of the vector $\hat{i}-2\hat{j}+2\hat{k}$ and passing through a point $2\hat{i}-2\hat{j}+2\hat{k}$. Find the moment of force about the point $\hat{i}+\hat{j}+\hat{k}$ [V. H-92, 94]

Ans: Let $\vec{F}$ be the force in the direction of vector $\hat{i}-2\hat{j}+2\hat{k}$ having magnitude 15 units.

$$\therefore \vec{F} = \frac{15(\hat{i}-2\hat{j}+2\hat{k})}{\sqrt{1+4+4}}$$

$$= \frac{15}{3}(\hat{i}-2\hat{j}+2\hat{k}).$$

The position vector of the point $(2\hat{i}-2\hat{j}+2\hat{k})$ relative $\hat{i}+\hat{j}+\hat{k}$ is.

$$\vec{r} = (2\hat{i}-2\hat{j}+2\hat{k}) - (\hat{i}+\hat{j}+\hat{k})$$

$$= \hat{i}-3\hat{j}+\hat{k}.$$

Hence, the moment of force is $\vec{r}\times\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 5 & -10 & 10 \end{vmatrix}$$

$$= \hat{i}(-30+10)+\hat{j}(5-10)+\hat{k}(-10+15)$$

$$= -20\hat{i}-5\hat{j}+5\hat{k}.$$

$\boxed{\text{Ex-7}}$: Show that the torque about the point $(2\hat{i}+\hat{j}-3\hat{k})$ of the force $(\hat{i}+2\hat{j}+\hat{k})$ passing throug the point $3\hat{i}+4\hat{j}-\hat{k}$ is $\hat{j}-\hat{i}-\hat{k}$.

Ans: Try yourself.

Ex-8: Find the moment of a force represented by $\vec{PQ}$ about an axis through the point $A(3,1,0)$ in the direction of the vector $2\hat{i}+3\hat{j}+6\hat{k}$ where the position vectors of P and Q are $\hat{i}+3\hat{j}+2\hat{k}$ and $3\hat{i}+4\hat{j}+3\hat{k}$ respectively.

Ans: The position vector of the point $P(\hat{i}+3\hat{j}+2\hat{k})$ relative to $A(3,1,0)$ is $\vec{r} = \vec{AP} = \vec{OP} - \vec{OA}$

$$= (\hat{i}+3\hat{j}+2\hat{k}) - (3\hat{i}+\hat{j})$$

$$= -2\hat{i}+2\hat{j}+2\hat{k}.$$

The force is $\vec{F} = \vec{PQ} = (3\hat{i}+4\hat{j}+3\hat{k}) - (\hat{i}+3\hat{j}+2\hat{k})$

$$= 2\hat{i}+\hat{j}+\hat{k}.$$

$\therefore$ Moment is $\vec{r}\times\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 6\hat{j} - 6\hat{k}.$$

The required moment in the direction of the vector $2\hat{i}+3\hat{j}+6\hat{k}$

is $(\vec{r}\times\vec{F}) \cdot \dfrac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$

$$= (6\hat{j}-6\hat{k}) \cdot \frac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}$$

$$= \frac{18-36}{7} = -\frac{18}{7}.$$

Ex-9: A rigid body is spinning with an angular velocity 5 radians/sec about an axis of direction $(0,3,-1)$ passing through the point $A(1,3,-1)$. Find the velocity of the particle at the point $P(4,-2,1)$.

Ans: The angular velocity $\vec{\omega}$ having magnitude 5 radian/sec. in the direction $(0,3,-1)$ is $\vec{\omega} = \dfrac{5(3\hat{j}-\hat{k})}{\sqrt{10}}$.

The position vector of P relative to A is

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$$\vec{r} = \vec{AP} = (4\hat{i} - 2\hat{j} + \hat{k}) - (\hat{i} + 3\hat{j} - \hat{k})$$
$$= 3\hat{i} - 5\hat{j} + 2\hat{k}.$$

Hence the required velocity $\vec{v}$ of the particle is

$$\vec{v} = \vec{\omega} \times \vec{r}$$
$$= \frac{5}{\sqrt{10}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & -1 \\ 3 & -5 & 2 \end{vmatrix}$$
$$= \frac{5}{\sqrt{10}} (\hat{i} - 3\hat{j} - 9\hat{k}).$$

This represents a speed $|\vec{\omega} \times \vec{r}| = \frac{5}{\sqrt{10}} |(\hat{i} - 3\hat{j} - 9\hat{k})|$.

$= 5\sqrt{\frac{91}{10}}$ in the direction of the vector $\hat{i} - 3\hat{j} - 9\hat{k}$.

$\boxed{\text{Ex-10}}$: The angular velocity of a rotating rigid body about an axis of rotation is given by $4\hat{i} + \hat{j} - 2\hat{k}$. Find the linear velocity of a point on the body whose position vector relative to a point on the axis of rotation is $2\hat{i} - 3\hat{j} + \hat{k}$.

<u>Ans</u>: Given that angular velocity $\vec{\omega} = 4\hat{i} + \hat{j} - 2\hat{k}$.

The linear velocity $\vec{v} = \vec{\omega} \times \vec{r}$

$$= (4\hat{i} + \hat{j} - 2\hat{k}) \times (2\hat{i} - 3\hat{j} + \hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$
$$= -5\hat{i} - 8\hat{j} - 14\hat{k}$$
$$= -(5\hat{i} + 8\hat{j} + 14\hat{k}).$$

$\boxed{\text{Ex-11}}$: Find the velocities of the particle at the points $(1, 2, 3)$ and $(1, 0, 4)$ of a rigid body which is spinning about a fixed point $(2,-1,-3)$ with angular velocity 5 radians/sec, the axis of rotation being in the direction of $(-2\hat{i}+\hat{j}+2\hat{k})$.

<u>Ans</u>: The angular velocity $\vec{\omega}$ having magnitude 5 radian/sec in the direction $(-2\hat{i}+\hat{j}+2\hat{k})$ is $\frac{5(-2\hat{i}+\hat{j}+2\hat{k})}{3}$.

The position vector of $P(1,2,3)$ relative to $A(2,-1,-3)$ is

$$\vec{r} = \vec{AP} = (\hat{i}+2\hat{j}+3\hat{k}) - (2\hat{i}-\hat{j}-3\hat{k})$$
$$= -\hat{i}+3\hat{j}+6\hat{k}.$$

Also, the position vector of $Q(1,0,4)$ relative to $A(2,-1,-3)$ is

$$\vec{r} = \vec{AQ} = (\hat{i}+4\hat{k}) - (2\hat{i}-\hat{j}-3\hat{k})$$
$$= -\hat{i}+\hat{j}+7\hat{k}.$$

$\therefore$ The velocity at $(1,2,3)$ is $\vec{\omega}\times\vec{AP}$

$$= \frac{5}{3}(-2\hat{i}+\hat{j}+2\hat{k}) \times (-\hat{i}+3\hat{j}+6\hat{k})$$

$$= \frac{5}{3}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 2 \\ -1 & 3 & 6 \end{vmatrix}$$

$$= \frac{5}{3}(10\hat{j}-5\hat{k})$$

$$\boxed{= \frac{25}{3}(2\hat{j}-\hat{k})}.$$

Also, the velocity of the particle at $(1,0,4)$ is

$$\vec{\omega}\times\vec{AQ}$$

$$= \frac{5}{3}(-2\hat{i}+\hat{j}+2\hat{k}) \times (-\hat{i}+\hat{j}+7\hat{k})$$

$$= \frac{5}{3}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 2 \\ -1 & 1 & 7 \end{vmatrix}$$

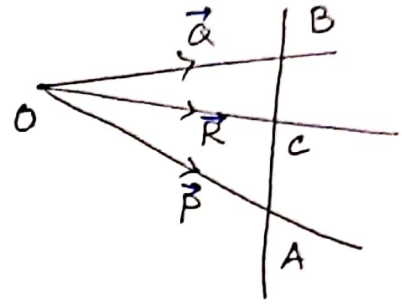
$$\boxed{= \frac{5}{3}(5\hat{i}+12\hat{j}-\hat{k})}.$$

Ex-12 : Forces $\vec{P}, \vec{Q}$ act at O and have a resultant $\vec{R}$. If any transversal cuts their lines of action at the points A, B, C respectively, then show that $\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$. V.H-'99

Ans : The given forces are $\vec{P}$ and $\vec{Q}$ and the resultant force is $\vec{R}$ $[\because |\vec{P}| = P, |\vec{Q}| = Q, |\vec{R}| = R]$

~~So the magnitude of $\vec{P}, \vec{Q}$ and $\vec{R}$ be P, Q, R resp.~~

Let the position vectors of the points A, B, C with respect to the point O be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.



So, the forces are $P\frac{\vec{a}}{|\vec{OA}|}$, $Q\frac{\vec{b}}{|\vec{OB}|}$ and $R\frac{\vec{c}}{|\vec{OC}|}$.

Since, $\vec{R}$ is the resultant of $\vec{P}$ and $\vec{Q}$.

$$\therefore \quad P\frac{\vec{a}}{|\vec{a}|} + Q\frac{\vec{b}}{|\vec{b}|} = R\frac{\vec{c}}{|\vec{c}|}$$

$[|\vec{OA}| = |\vec{a}| = OA, |\vec{OB}| = |\vec{b}| = OB$ and $|\vec{OC}| = |\vec{c}| = OC]$

$$\Rightarrow \frac{P}{OA}\vec{a} + \frac{Q}{OB}\vec{b} - \frac{R}{OC}\vec{c} = 0.$$

Since, A, B, C are collinear, we must have the sum of the coefficients in this relation equal to zero.

$$\therefore \quad \frac{P}{OA} + \frac{Q}{OB} - \frac{R}{OC} = 0$$

$$\Rightarrow \frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC} \quad \text{(Proved)}$$